D–Branes and Fluxes in Supersymmetric Quantum Mechanics

James E. Carlisle[♭], **Clifford V. Johnson**[♮,1], **Jeffrey S. Pennington**[♯]

[♭] *Centre for Particle Theory*
Department of Mathematical Sciences
University of Durham
Durham DH1 3LE, England, U.K.

[♮,♯] *Department of Physics and Astronomy*
University of Southern California
Los Angeles, CA 90089-0484, U.S.A.

j.e.carlisle@durham.ac.uk, johnson1@usc.edu, jspennin@usc.edu

Abstract

Type 0A string theory in the $(2, 4k)$ superconformal minimal model backgrounds, with background ZZ D–branes or R–R fluxes can be formulated non–perturbatively. The branes and fluxes have a description as threshold bound states in an associated one–dimensional quantum mechanics which has a supersymmetric structure, familiar from studies of the generalized KdV system. The relevant bound state wavefunctions in this problem have unusual asymptotics (they are not normalizable in general, and break supersymmetry) which are consistent with the underlying description in terms of open and closed string sectors. The overall organization of the physics is very pleasing: The physics of the closed strings in the background of branes or fluxes is captured by the generalized KdV system and non–perturbative string equations obtained by reduction of that system (the hierarchy of equations found by Dalley, Johnson, Morris and Wätterstam). Meanwhile, the bound states wavefunctions, which describe the physics of the ZZ D–brane (or flux) background in interaction with probe FZZT D–branes, are captured by the generalized mKdV system, and non–perturbative string equations obtained by reduction of that system (the Painlevé II hierachy found by Periwal and Shevitz in this context).

[1] Also, Visiting Professor at the Centre for Particle Theory[♭].

1 Opening and Closing Remarks

It is safe to say that, at this point in time, we do not understand string (or M–) theory as well as we would like. While we have understood and appreciated that there is a rich bounty of physical phenomena contained in the theory, this has mostly been uncovered in perturbation theory, occasionally sweetened by a glimpse into the non–perturbative realm afforded by special sectors of the theory such as soliton solutions (including branes of various sorts) or various topological reductions.

The physics that we have so far learned from the theory has provided numerous promising and exciting phenomenological scenarios that form the basis for several research endeavours to understand and incorporate current experimental and observational data from Nature, and furnish testable predictions about new physics. These endeavours are still embryonic, and cannot fully mature without much more understanding of the underlying theory.

Furthermore, much of what we have learned pertains to the critical string theories, a rich class for study of course, but after all of the non–perturbative lessons that we have learned in the last decade, the fact that as a field we mostly still linger in the critical domain should be regarded as nothing more than the force of habit; so much historical baggage. Having broken free of the shackles of perturbative thinking, there is no compelling physical reason to restrict attention to critical strings in a search for a description of Nature. It is time to try to move on to other areas of the theory, where the tools and concepts we need to make contact with Nature may well be waiting to be found.

There has been some movement. Due to progress in the understanding of open string sectors in Liouville conformal field theory[1, 2, 3], and following on from the proposal in ref.[4], recent years have seen a growing realisation that the c (or $\hat{c} \leq 1$) non-critical string theories (equivalently $D \leq 2$, so "subcritical"), despite being rather simple as compared to their higher dimensional cousins, contain several model examples of the non–perturbative phenomena that have so fascinated us from higher dimensional critical strings such as D–branes, holography, open–closed transitions, tachyon condensation, etc. In fact, this class of models —first arrived at by double scaling certain matrix models[5, 6, 7, 8]— contains the earliest examples of fully non–perturbative formulations of string theories, which remain the *only* formulations available where one can ask and answer (appropriate) questions arbitrarily far from perturbation theory. Furthermore, the fact that one can get different string theories by expanding the physics in different small parameters (something we'd like to better understand about M–theory and the critical string theories) is manifest in these models. For example, in one class of models

first found and studied extensively in refs.[9, 10, 11, 12], identified as $\hat{c} < 1$ type 0A strings in refs.[13], and to be further discussed at length in this paper, the physics is contained rather succinctly in a non–linear differential equation, with no reference to strings and their world–sheets. It is only when a small dimensionless parameter is identified and the solution is expanded in terms of this parameter does the physics take on the interpretation of a string theory (where the small parameter is the string coupling) which can be open or closed depending upon which parameter is taken to be small[1].

The celebrated non–perturbative phenomena mentioned near the beginning of the previous paragraph are examples of exciting physics of which we would like even more examples, and of which we would like better understanding. The type of non–perturbative formulations under discussion furnish such examples and enhance our understanding somewhat by sharpening the terms in which the phenomena of interest are expressed and by confirming them as robust (perhaps even generic) non–perturbative features of the theory.

Double scaled matrix models (and their accompanying physics) were abandoned as non–perturbative approaches by the field only a few years after their first construction, the main reasons cited being non–perturbative ambiguities (see e.g., discussion and analysis in refs.[15, 16, 17]) and oversimplicity. This was despite clear demonstrations that there were fully consistent and unambiguous models[9, 10, 18, 19, 20] available which avoided these objections, and non–perturbative maps between models with closed and open strings[12]. We should be careful to not make the same mistake twice and again turn our attention away from these models prematurely. There is an important question to ask: Now that we have recognised that these models describe so many of our favourite important non–perturbative phenomena, can we learn from them about new non–perturbative physics that has hitherto been overlooked?

It is with this question in mind that we continue our investigations in this area. In this paper we explore further a number of the observations reported in our previous paper[21]. We sharpen the observations and explicitly extend several features. A particular theme which has arisen, and that is quite striking in the results of this paper and the last is the fact that the underlying connections to structures from certain integrable systems seem to be becoming physically clearer and broader in scope. For example, while it has been known for some time[8, 22] that the Korteweg–de Vries (KdV) hierarchy of flows organises the closed string operator content in these models, it was only recently realised (in our previous paper[21]) that the well–known Bäcklund transformations that change the number of solitons in a solution of the KdV equation actually

[1] For some of the $D = 2$ strings in this class, there has even been a recent attempt to formulate a sort of M–theory specifically. See ref.[14].

have meaning in this context: They change the number of background "ZZ"[2] D–branes in the model, in one regime, or the number of units of background R–R flux in another. That there is a one–to–one correspondence between D–branes (usually thought of as solitons in a very different sense) and solitons of KdV (the prototype solitons) is both ironic and interesting. We will explore this and the role of the Bäcklund transformation further in this paper. Closely allied to the Bäcklund transformation is the Miura map which connects solutions of the KdV hierarchy to that of another integrable hierarchy, the "modified" KdV (mKdV) system. It was noticed long ago in ref.[18] that this map is invertible when applied to solutions of the string equations for (what we now know is) the type 0A system, and connects them to solutions of another set of equations, the Painlevé II hierarchy. These equations were known (from the work of Periwal and Shevitz[23, 24]) to arise from double scaling unitary matrix models, but it was not understood exactly what was their string theory interpretation. The work of ref.[18] therefore gave an interpretation, for the first time, for the double scaled unitary matrix models: They were really secretly systems of open and closed strings, just written in rather apparently strange variables defined by the Miura map. Now we know that there is more, and in this paper we make explicit the role of the unitary matrix models and their associated Painlevé II string equations. The physics derived from those systems is simply that of a probe "FZZT"[3, 1, 2] D–brane when it has been stretched entirely in the Liouville direction, terminating on a family of ZZ D–branes living at positive infinity. The worldsheet expansions obtained from the solution of the Painlevé II equation give all of the open string worldsheets associated to this configuration.

We show that the partition functions of this special probe configuration are actually a type of non–normalizable threshold bound state wavefunction of an associated supersymmetric quantum mechanics (a system well–known to be associated to the KdV–mKdV hierarchies), and their non–normalizability serves to spontaneously break the supersymmetry. These particular wavefunctions are a special case of the Baker–Akheizer function of the integrable system, already identified[13] as being the partition function of FZZT D–brane probes.

This sharper understanding of the interconnected role of the KdV, mKdV, Bäcklund, Miura, Baker–Akheizer structures in terms of familiar physical objects in open and closed string theory is rather pleasing, and highly suggestive.

2 A Quantum Mechanics Problem and a String Theory

Consider the following one–dimensional quantum mechanics problem:

$$\mathcal{H}\psi(z) = \lambda\psi(z) \ , \tag{1}$$

where the Hamiltonian is:

$$\mathcal{H} = -Q = -\nu^2\frac{\partial^2}{\partial z^2} + u(z) = -d^2 + u \ . \tag{2}$$

The potential $u(z)$ satisfies the differential equation[12]:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}'' + \frac{1}{4}(\mathcal{R}')^2 = \nu^2\Gamma^2 \ . \tag{3}$$

and here $\mathcal{R} = u(z) - z$. Our physics will require that $u(z)$ is a real function of the real variable z. A prime denotes $\nu\partial/\partial z$, and Γ and ν are constants. The physics also imposes the following boundary conditions:

$$\begin{aligned} u &\rightarrow z + O(\nu); \qquad \text{as} \quad z \rightarrow +\infty \\ u &\rightarrow 0 + O(\nu^2); \qquad \text{as} \quad z \rightarrow -\infty \ . \end{aligned} \tag{4}$$

Evidently ν plays the role of $\hbar$ in this system. The classical limit ($\hbar \rightarrow 0$) will be the classical limit of the underlying string theory, as we shall see: Taking a few more terms in the expansion of $u(z)$, we have:

$$\begin{aligned} u &\rightarrow z + \frac{\nu\Gamma}{z^{1/2}} - \frac{\nu^2\Gamma^2}{2z^2} + \frac{5}{32}\frac{\nu^3}{z^{7/2}}\Gamma(4\Gamma^2+1)\cdots; \qquad \text{as} \quad z \rightarrow +\infty \\ u &\rightarrow 0 + \frac{\nu^2(4\Gamma^2-1)}{4z^2} + \frac{\nu^4}{8}\frac{(4\Gamma^2-1)(4\Gamma^2-9)}{z^5}\cdots; \qquad \text{as} \quad z \rightarrow -\infty \ . \end{aligned} \tag{5}$$

The partition function $Z = \exp(-F)$ of the string theory is obtained from $u(z)$ using:

$$u(z) = \nu^2\frac{\partial^2 F}{\partial\mu^2}\bigg|_{\mu=z} \ , \tag{6}$$

where μ is the coefficient of the lowest dimension operator in the world–sheet theory. So integrating twice, we get an expansion:

$$F = \frac{1}{6}g_s^{-2} + \frac{4}{3}\Gamma g_s^{-1} + \frac{1}{2}\Gamma^2 g_s^0 \ln\mu + \frac{1}{24}\Gamma(4\Gamma^2+1)g_s^1 + \cdots; \qquad \text{as} \quad z \rightarrow +\infty \tag{7}$$

$$F = -\left(\Gamma^2 - \frac{1}{4}\right)g_s^0\log\mu + \frac{1}{96}(4\Gamma^2-1)(4\Gamma^2-9)g_s^2 + \cdots; \qquad \text{as} \quad z \rightarrow -\infty \ , \tag{8}$$

which is an asymptotic expansion in the dimensionless string coupling $g_s = \nu/\mu^{3/2}$. In this expansion, since the sphere term is non–universal and can be dropped (since it contains no non–analytic behaviour in μ), the positive z region is a purely open string expansion, and Γ, which comes multiplying every worldsheet boundary, has the interpretation as counting the number of species of D–brane present[12]. These are fully localized "ZZ" D–branes. See figure 1.

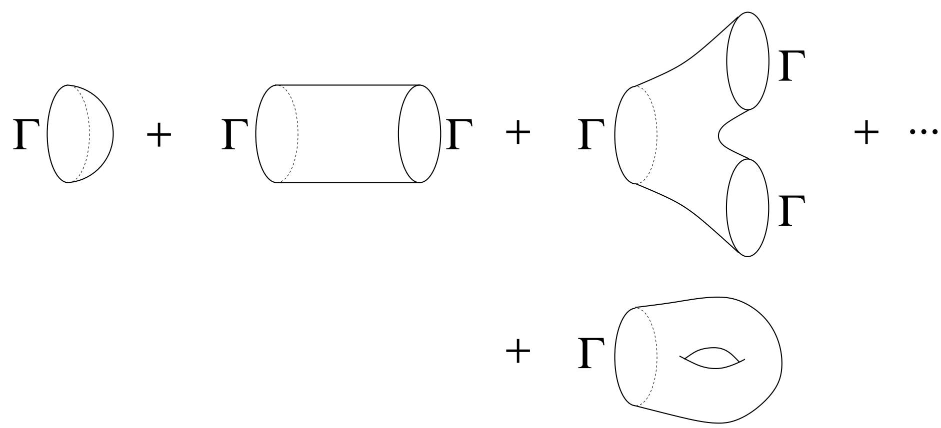


Figure 1: Some diagrams contributing to the worldsheet expansion (7) given by perturbatively solving the string equation for large positive z.

The negative z region is a purely closed string expansion, and Γg_s accompanies a worldsheet insertion of a vertex operator in the presence of Γ units of background R–R flux. See figure 2. Each vertex operator insertion gives a factor $g_s\Gamma$. There must be an even number of insertions, as shown by a worldsheet computation in the continuum theory[13].

Let us look at the potential. Figure 3 shows the typical features, discussed in detail in ref.[21]. In figure 3*(a)* we show only the case of positive integer Γ, while in figure 3*(b)* we show the case of $-1 < \Gamma < 0$. Note that it becomes progressively more difficult to solve for $u(z)$ with the given boundary conditions as Γ approaches minus one. However the Bäcklund transformation (defined later in equation (9)) defined below allows us to overcome this difficulty, and the results are displayed in figure 4.

In ref.[21] we showed, using a combination of numerical and analytic studies, that the case of integer Γ is rather distinct from that of non–integer Γ. Furthermore, we showed there that once one restricts to the integers, the positive integers are selected by the system, since there is a non–perturbative transformation which relates u_Γ and $u_{\Gamma\pm1}$ (which was first deduced in

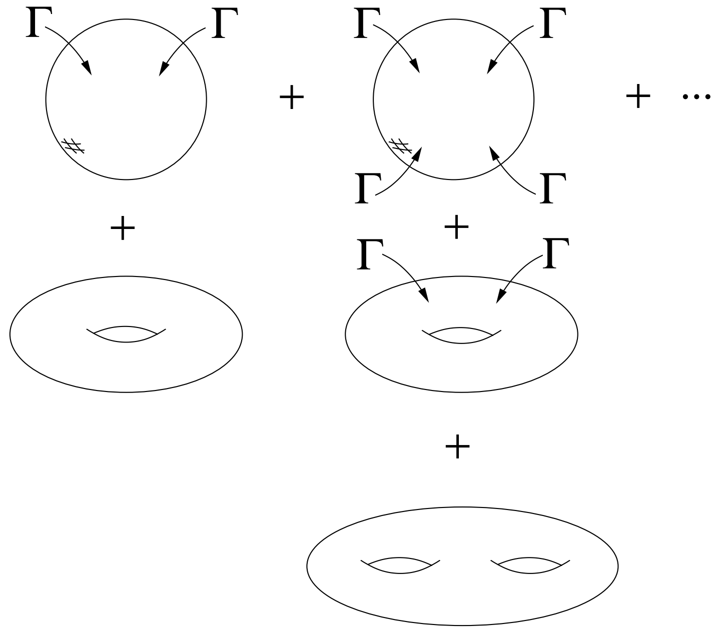


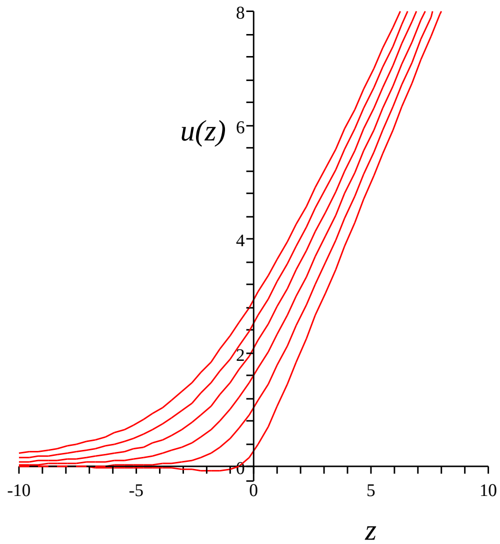
Figure 2: Some diagrams contributing to the worldsheet expansion (8) given by perturbatively solving the string equation for large negative z. Each vertex operator insertion gives a factor $g_s\Gamma$. There must be an even number of insertions.

ref.[12], and made explicit in ref.[21]):

$$u_{\Gamma\pm1} = \frac{3\,(\mathcal{R}')^2 - 2\mathcal{R}\mathcal{R}'' \mp 8\nu\Gamma\,\mathcal{R}' + 4\nu^2\Gamma^2}{4\mathcal{R}^2}\ , \tag{9}$$

where $\mathcal{R} \equiv \mathcal{R}(u_\Gamma)$, and starting with $u_{\Gamma=0}$ it is easy to see that $u_{\Gamma=1} = u_{\Gamma=-1}$, as is suggested[21] in the numerical behaviour of the function for arbitrary Γ studies (see figure 4). By extension, it is clear that $u_\Gamma = u_{-\Gamma}$. This is consistent with the theory being charge conjugation invariant. In fact, one can run the argument the other way around: Starting with a requirement that the theory be charge conjugation invariant, the properties of the equations and their solutions that we uncovered are enough to prove that Γ must be a positive integer. It was also noticed in ref. [21] that the transformation above is in fact the celebrated Bäcklund transformation of the KdV system, specialized to our system of solutions.

It was also realized in ref.[21] that Γ represents the formal soliton number of the solution given by $u(z)$, further reinforcing the idea that the positive integers are natural values for it to take. This is why the Bäcklund transformation naturally acts here. They are well–known to change by an integer the number of solitons in a KdV solution. We use the term "formal" above because $u(z)$ does not have standard soliton boundary conditions (it does not vanish in both asymptotic directions), and because the bound states to which the solitons correspond (*via*



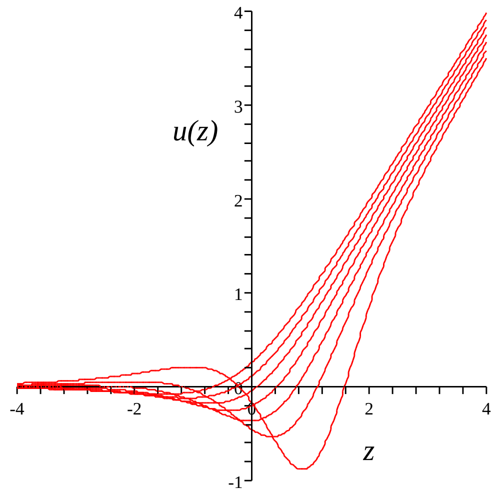


Figure 3: Numerical solutions to equation (3) for $u(z)$: *(a)* cases of positive integer Γ, and *(b)* some cases of $-1 < \Gamma < 0$.

inverse scattering theory) are of zero energy ($\lambda = 0$ in equation (1)).

Let us now develop this much further. Turning to study the properties of our wavefunctions of this system, we can make the following observations. The potential becomes linear in the large positive z regime, $u(z) = z + \ldots$, and so the wavefunction will behave like an Airy function in this regime:

$$\psi(z) = (z+\lambda)^{-\frac{1}{4}} e^{\pm\frac{2}{3}(z+\lambda)^{\frac{3}{2}}} + \cdots \tag{10}$$

Meanwhile, in the large negative z regime, the potential vanishes to leading order, and at next order is:

$$u(z) = \frac{\nu^2}{z^2}\left(\Gamma^2 - \frac{1}{4}\right) + \cdots \tag{11}$$

so the Schrödinger equation is:

$$-\nu^2 \frac{d^2\psi(z)}{dz^2} + \frac{\nu^2}{z^2}\left(\Gamma^2 - \frac{1}{4}\right)\psi(z) = \lambda\psi(z) \ . \tag{12}$$

Notice what happens when we change variables using $\psi(z) = z^{1/2}\phi(z)$, and define $x = \lambda^{1/2} z/\nu$, We get the equation:

$$x^2 \frac{d^2\phi(x)}{dx^2} + x\frac{d\phi(x)}{dx} + \left(x^2 - \Gamma^2\right)\phi(x) = 0 \ , \tag{13}$$

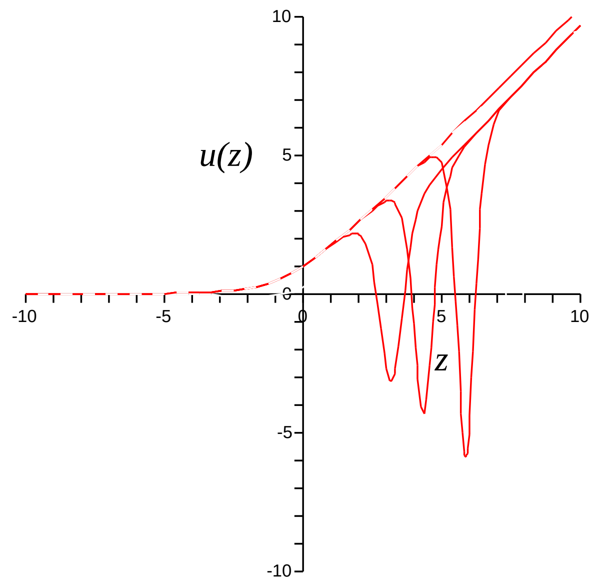


Figure 4: Three numerical examples $u(z)$ of $\Gamma = -1 + \epsilon$ for ϵ small and positive, generated by using the Bäcklund transformations on $\Gamma = \epsilon$ curves. They increasingly approach the $\Gamma = +1$ curve (shown dotted at the top), in a manner which is consistent with a limit where the exact identity $u(z)_{\Gamma=-1} = u(z)_{\Gamma=+1}$ is satisfied (see text).

which is simply Bessel's equation, whose natural solutions are the Bessel functions, and we choose the modified variety $\phi(x) = I_\Gamma(x)$, since $\psi(z)$ should be real. The behaviour (for any Γ) in this regime is, after converting back to the original variables is:

$$\psi(z) = e^{\frac{\lambda^{\frac{1}{2}}}{\nu} z} + \cdots \tag{14}$$

We wish to focus on the case of vanishing energy, λ, which is associated with the Bäcklund transformation between u_Γ and $u_{\Gamma\pm1}$. This will lead to an understanding of how $\psi(z)$ encodes the physics of our background D–branes and fluxes. To proceed we factorize the Hamiltonian as:

$$(\mathcal{H}_\Gamma - \lambda)\,\psi = \left(-d^2 + u_\Gamma - \lambda\right)\psi = \left[(-d+v)\,(d+v) - \lambda\right]\psi = 0 \ , \tag{15}$$

where

$$v^2 - v' = u_\Gamma \ , \tag{16}$$

The continuum begins at $\lambda = 0$, corresponding to threshold bound states of zero energy. Specializing to that case, we have $\psi' = -v\psi$, and so we deduce that (up to a normalization)

we can always write our wavefunctions as:

$$\psi = e^{-\int v/\nu} \ , \tag{17}$$

and ask what equation v satisfies. Some algebra shows that if u_Γ, satisfies (3), then v satisfies:

$$\frac{1}{2}v'' - v^3 + zv + \nu C = 0 \ , \quad \text{where} \quad C = \frac{1}{2} \pm \Gamma \ . \tag{18}$$

This is the Painlevé II equation. Actually, there are two choices for the constant C, for every Γ, which we'll sometimes write as C_+ and C_-, although we will sometimes also use the fact that given either choice for C, the other is $1 - C$. This gives two cases of interest for every choice of Γ, and so we shall sometimes call these v_Γ and $\bar{v}_\Gamma$, respectively:

$$\begin{aligned}
v_\Gamma \ &\rightarrow \ z^{1/2} + \frac{1}{2}\frac{\nu C_+}{z} - \frac{1}{32}\frac{\nu^2}{z^{5/2}}(12\Gamma^2 + 12\Gamma + 5) + \cdots \ ; \qquad \text{as} \quad z \rightarrow +\infty \\
v_\Gamma \ &\rightarrow \ -\frac{\nu C_+}{z} - \frac{1}{8}\frac{\nu^3}{z^4}(4\Gamma^2 - 1)(2\Gamma + 3) + \cdots \ ; \qquad \text{as} \quad z \rightarrow -\infty \\
\bar{v}_\Gamma \ &\rightarrow \ -z^{1/2} + \frac{1}{2}\frac{\nu C_-}{z} + \frac{1}{32}\frac{\nu^2}{z^{5/2}}(12\Gamma^2 - 12\Gamma + 5) + \cdots \ ; \qquad \text{as} \quad z \rightarrow +\infty \\
\bar{v}_\Gamma \ &\rightarrow \ -\frac{\nu C_-}{z} + \frac{1}{8}\frac{\nu^3}{z^4}(4\Gamma^2 - 1)(2\Gamma - 3) + \cdots \ ; \qquad \text{as} \quad z \rightarrow -\infty \ ,
\end{aligned} \tag{19}$$

and so integrating and exponentiating to form the wavefunction we find that there are two choices for wavefunctions:

$$\begin{aligned}
\psi_\Gamma \ &\rightarrow \ z^{-\frac{1}{2}(\frac{1}{2}+\Gamma)}e^{+\frac{2}{3}z^{3/2}} + \cdots \ ; \qquad \text{as} \quad z \rightarrow +\infty \\
\psi_\Gamma \ &\rightarrow \ z^{-\frac{1}{2}-\Gamma} + \cdots \ ; \qquad \text{as} \quad z \rightarrow -\infty \\
\bar{\psi}_\Gamma \ &\rightarrow \ z^{\frac{1}{2}(\frac{1}{2}-\Gamma)}e^{-\frac{2}{3}z^{3/2}} + \cdots \ ; \qquad \text{as} \quad z \rightarrow +\infty \\
\bar{\psi}_\Gamma \ &\rightarrow \ z^{-\frac{1}{2}+\Gamma} + \cdots \ ; \qquad \text{as} \quad z \rightarrow -\infty \ ,
\end{aligned} \tag{20}$$

This is appropriate, since the potential $u(z) \rightarrow z$ as $z \rightarrow +\infty$ and so the wavefunction should resemble the exponential tail of the Airy function in that limit, and it does. Meanwhile in the $z \rightarrow -\infty$ limit, where the potential vanishes to leading order, the wavefunction has a purely power law behaviour, entirely appropriate for a zero energy state. Rather than have Γ different wavefunctions for the Γ different objects of degenerate energy (which one ought not to expect in one dimensional quantum mechanics) $\psi(z)$ and $\bar{\psi}(z)$ should be thought of as describing Γ bound objects, signalled by the non–exponential part of their behaviour as $z \rightarrow \pm\infty$. Shortly, we will further unpack the meaning of $\psi(z)$ and $\bar{\psi}(z)$.

The form of $v_\Gamma(z)$, $\bar{v}_\Gamma(z)$ and the associated wavefunctions $\psi_\Gamma(z)$ and $\bar{\psi}_\Gamma(z)$ (recovered *via* equation (17)) can be exhibited numerically, and examples are given in figure 5 (for $v(z)$), 6

(for $\bar{v}(z)$), and figure 7, (for $\psi(z)$ and $\bar{\psi}(z)$). Note that it becomes progressively more difficult to solve for $v(z)$ ($\bar{v}(z)$) with the given boundary conditions as Γ approaches minus one (zero). However, we will later define Bäcklund transformations (61) for $v(z)$ and $\bar{v}(z)$ that allow us to overcome this difficulty. We have made use of these transformations in figures 5*(b)* and 6*(b)*.

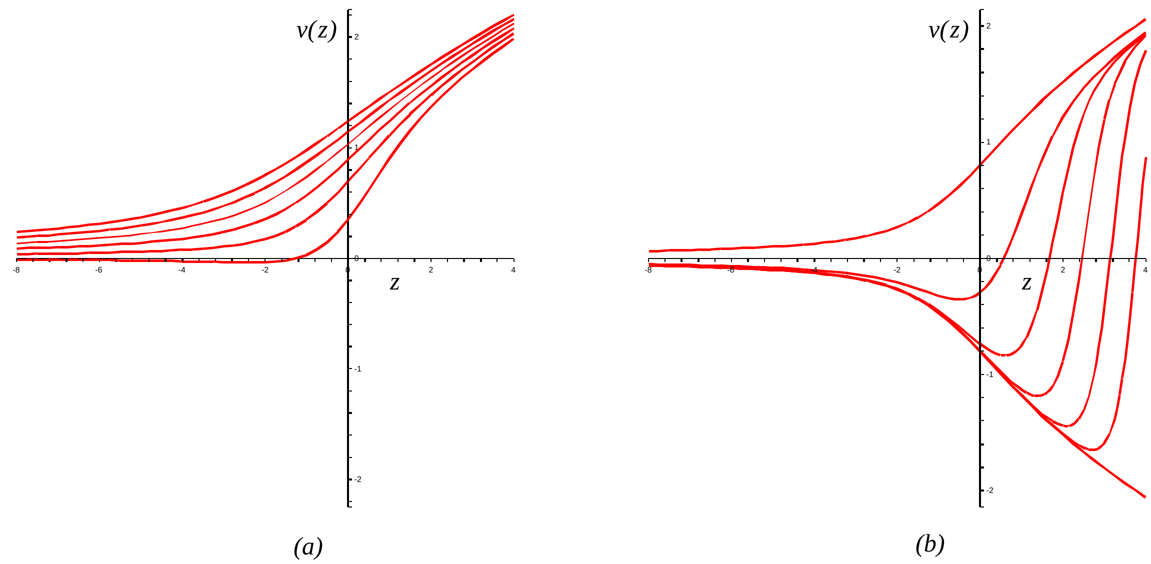


Figure 5: Numerical solutions for $v(z)$ for a range of values of Γ. From top to bottom, we have *(a)*: $\Gamma = 1.4, 1.0, 0.6, 0.2, -0.2, -0.6$, and *(b)*: $\Gamma = 0.0, -0.9, -0.99, -0.999, -0.9999, -0.99999$ generated by the Bäcklund transformations (61), ending with the curve of $\bar{v}(z)$ for $\Gamma = 1$ (see later in text for explanation).

Notice that none of the wavefunctions are normalizable, regardless of the value of Γ. Note also that the case $\Gamma = 0$ has $\psi \sim z^{-1/2}$ to the left and $\psi \sim z^{1/4} \exp(\pm\frac{2}{3}z^{3/2})$, so we get a logarithmically divergent result for its integrated square. In a sense, this wavefunction is just on the cusp of normalizability. Finally, note that for $\Gamma = -\frac{1}{2}$, the wavefunction decays as $\psi(z) \sim z^{\frac{1}{2}} e^{-\frac{2}{3}z^{2/3}}$ for large positive z, and $\psi(z) \sim$ constant for large negative z with similar behaviour for $\bar{\psi}(z)$ when $\Gamma = +\frac{1}{2}$.

3 Supersymmetry and Bäcklund Transformations

The factorization process of the previous section

$$\mathcal{H}_\Gamma = \mathcal{A}^\dagger \mathcal{A} = -d^2 + u_\Gamma \ , \tag{21}$$

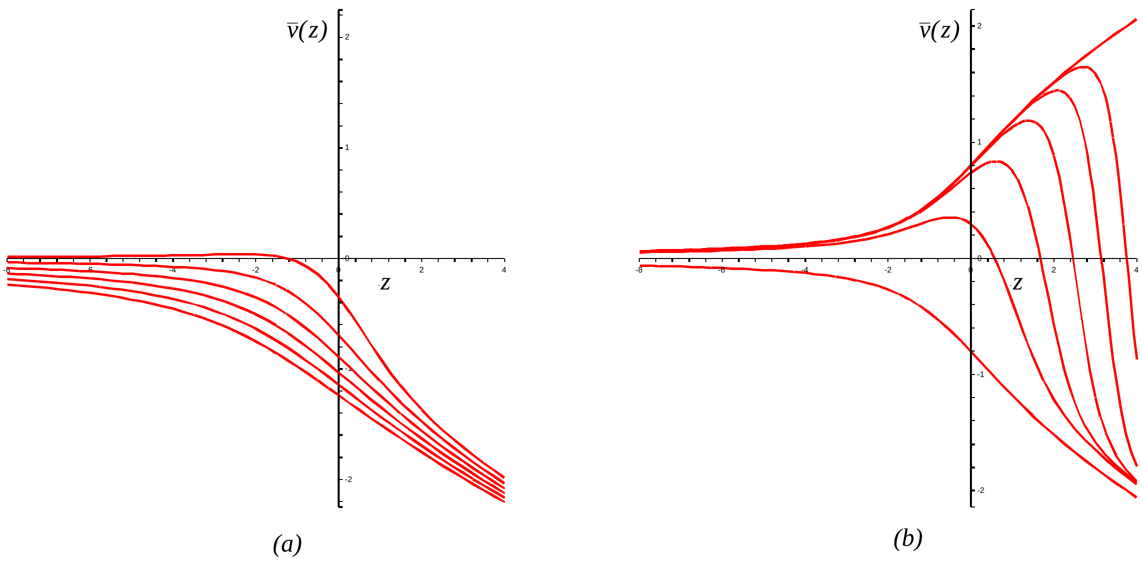


Figure 6: Numerical solutions for $\bar{v}(z)$ for a range of values of Γ. From bottom to top, we have *(a)*: $\Gamma = 2.4, 2.0, 1.6, 1.2, 0.8, 0.4,$ and *(b)*: $\Gamma = 1.0, 0.1, 0.01, 0.001, 0.0001, 0.00001$ generated by the Bäcklund transformations (61), ending with the curve of $v(z)$ for $\Gamma = 0$ (see later in text for explanation).

where $\mathcal{A} = d + v$ and $\mathcal{A}^\dagger = -d + v$ is the first step in constructing a supersymmetric structure. A "superpartner" Hamiltonian can be constructed for $\mathcal{H}_{\Gamma+1}$ by simply reversing the order of the factors, to form:

$$\mathcal{H}_{\Gamma+1} = \mathcal{A}\mathcal{A}^\dagger = -d^2 + u_{\Gamma+1} \quad , \tag{22}$$

where we have changed the label on $\mathcal{H}$ to reflect the fact that we now have a new potential, which is:

$$u_{\Gamma+1} = v^2 + v' \ . \tag{23}$$

It is easy to see that Γ has increased by unity. It follows from the fact that the Painlevé II equation (18) is unchanged under $v \to -v$ and $C \to -C$. Therefore, given a solution v to the equation, one can generate a new solution, $\tilde{v} = -v$ to a new Painleve II equation whose constant is $\tilde{C} = -C$ instead of C. By the Miura map (16), this defines a new solution $\tilde{u}$ to the string equation (3), but with a different value of the constant on the right hand side, which we can call $\tilde{\Gamma}$. We can work out what the value of $\tilde{\Gamma}$ is by writing $\tilde{C} = \frac{1}{2} \pm \tilde{\Gamma}$. Since $C = \frac{1}{2} + \Gamma$, $\tilde{C} = -\frac{1}{2} - \Gamma = \frac{1}{2} - (\Gamma + 1)$. In other words, $\tilde{u} = u_{\Gamma+1}$, and the sign flip on Painlevé II allows us to construct $u_{\Gamma+1}$ from the $\bar{v}_{\Gamma+1}$ function. One can eliminate v from the above formalism to obtain a Bäcklund tranformation relating u_Γ and $u_{\Gamma+1}$ [21].

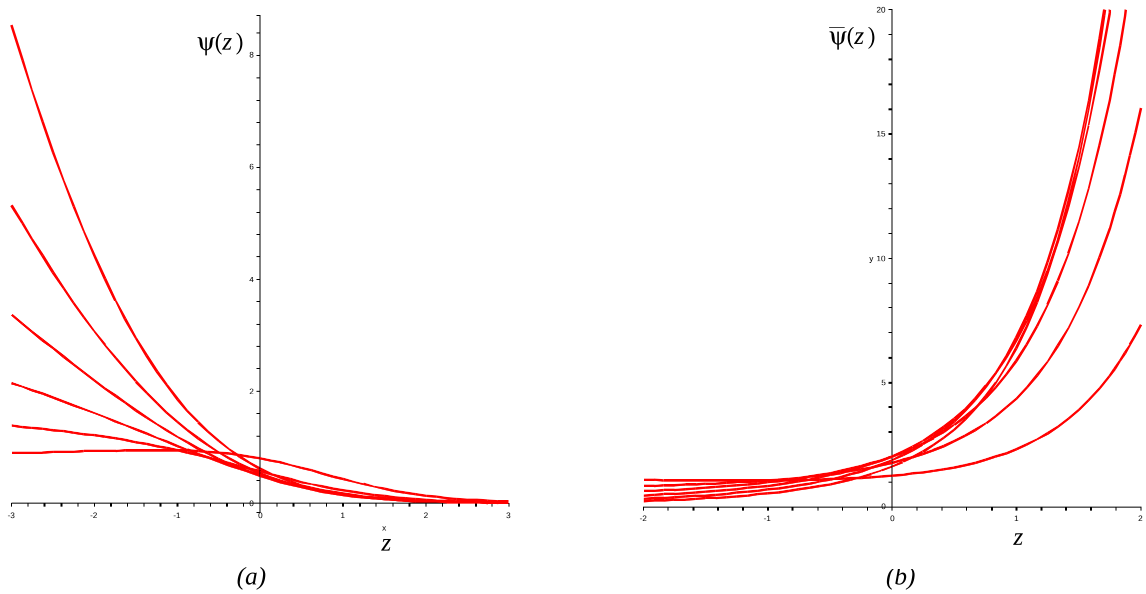


Figure 7: Numerical solutions for the wavefunctions $\psi(z)$ and $\bar{\psi}(z)$ for a range of values of Γ *(a)*: Top to bottom, $\Gamma = 1.4, 1.0, 0.6, 0.2, -0.2, -0.6$; *(b)*: Top to bottom, $\Gamma = 2.4, 2.0, 1.6, 1.2, 0.8, 0.4$

It is amusing to note that we can also construct $\mathcal{H}_{\Gamma-1}$ by instead choosing to do the sign flip on the $\bar{v}_\Gamma$ function. This gives us the equation $\tilde{C} = -(1-C) = \frac{1}{2} + (\Gamma - 1)$, telling us that the function $\tilde{u} = \bar{v}^2 + \bar{v}'$ is in fact $u_{\Gamma-1}$, now constructed from the $v_{\Gamma-1}$ funtion

$$\mathcal{H}_{\Gamma-1} = -d^2 + \bar{v}^2 + \bar{v} \ . \tag{24}$$

In this way we see that we can increase or decrease Γ by an integer depending upon whether we act with the sign flip on v_Γ or $\bar{v}_\Gamma$ (respectively), and the resulting $u_{\Gamma+1}$ or $u_{\Gamma-1}$ will be constructed from $\bar{v}_{\Gamma+1}$ and $v_{\Gamma-1}$, respectively. The structure is illustrated in figure 8.

Using this re–factorization to construct the Hamiltonian $\mathcal{H}_{\Gamma\pm1}$, we can now note that the spectra of the Hamiltonians of any two neighbouring Γ are related. If $\psi(z)$ is a wavefunction of $\mathcal{H}_\Gamma$ with *non–zero* eigenvalue λ, then by multiplying the eigenvalue equation on both sides by $\mathcal{A}$, one can see that it maps under "supersymmetry" to a wavefunction $\mathcal{A}\psi(z)$ of $\mathcal{H}_{\Gamma\pm1}$ with the *same* energy λ. Away from the zero energy sector therefore, $\mathcal{H}_\Gamma$ and $\mathcal{H}_{\Gamma\pm1}$ have identical spectra. Now there are no states for $\lambda < 0$, (see ref.[21]) and so we learn that the continuum parts of the spectrum are identical for each Hamiltonian. At $\lambda = 0$, the story is slightly different, however. There, the map fails, and we find that as we map the spectrum from $\mathcal{H}_\Gamma$ to that of $\mathcal{H}_{\Gamma+1}$, any $\lambda = 0$ state is lost.

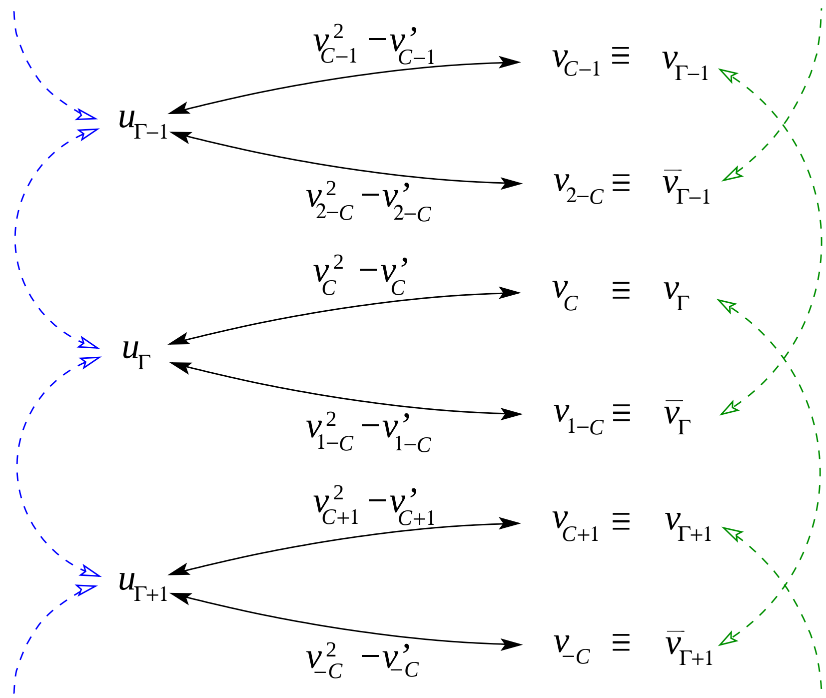


Figure 8: The structure of how the Miura map in combination with a sign flip on Painlevé II induces the Bäcklund transformations between u_Γ and $u_{\Gamma\pm1}$.

In the standard nomenclature, we can think of $\mathcal{H}_\Gamma$ as bosonic and $\mathcal{H}_{\Gamma\pm1}$ as fermionic, and we have such a pair for any value of Γ. A most efficient way of writing all of this to see the supersymmetric structure is to define the identity matrix, σ_0, together with the Pauli matrices, σ_j, $j = 1, 2, 3$:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{25}$$

(where here i is the square root of -1) and combine $\mathcal{H}_\Gamma$ and $\mathcal{H}_{\Gamma+1}$ into a larger Hamiltonian H:

$$H = (-d^2 + v^2)\sigma_0 - v'\sigma_3 = \begin{pmatrix} -d^2 + v^2 - v' & 0 \\ 0 & -d^2 + v^2 + v' \end{pmatrix} \ . \tag{26}$$

Defining $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, so that $\{\sigma_+, \sigma_-\} = 1$ and $[\sigma_+, \sigma_-] = -\sigma_3$, we can define the supercharges as:

$$Q^\dagger = -i\mathcal{A}^\dagger\sigma_- = i\begin{pmatrix} 0 & 0 \\ -d + v & 0 \end{pmatrix} \ , \qquad Q = -i\mathcal{A}\sigma_+ = i\begin{pmatrix} 0 & d + v \\ 0 & 0 \end{pmatrix} \ , \tag{27}$$

(in the last two displayed equations v can mean v_Γ or $\bar{v}_\Gamma$, depending upon whether going from $\mathcal{H}_\Gamma$ to $\mathcal{H}_{\Gamma+1}$ or to $\mathcal{H}_{\Gamma-1}$) so that we have the supersymmetry algebra:

$$\{Q^\dagger, Q\} = 2H \ , \qquad [Q, H] = 0 \ . \tag{28}$$

The condition for supersymmetry for a particular potential $u(z)$ and its superpartner potential is that we have a normalizable zero energy ground state, and we have already seen that for our system's boundary conditions, this fails. Our family of potentials therefore breaks supersymmetry.

4 The Physics of the Wavefunctions

4.1 The FZZT D-Brane as a Probe

So we've seen that we have a system with a non–normalizable wavefunctions, $\psi(z)$, $\bar{\psi}(z)$ for a state of zero energy, λ, which we nevertheless keep as part of the physics. The question naturally arises as to what the meaning of $\psi(z)$ and $\bar{\psi}(z)$ might be, and to what physics its properties described in the last section correspond. Somehow it encodes the information about the presence of Γ background objects. As learned in the context of AdS/CFT, the non–normalizability represents our having added something to the background: In this case, Γ D–branes, or units of R–R flux. How this is encoded in $\psi(z)$ and $\bar{\psi}(z)$ comes initially from expanding $v_{C_\pm}$. Let us start in the $z\to+\infty$ limit (recall $v_{C_+}\equiv v_\Gamma$, $v_{C_-}\equiv\bar{v}_\Gamma$), where from equation (19):

$$v_{C_\pm}=\pm z^{\frac{1}{2}}+\frac{1}{2}\frac{\left(\frac{1}{2}\pm\Gamma\right)}{z}\mp\frac{1}{32}\frac{\nu^2}{z^{5/2}}(12\Gamma^2\pm12\Gamma+5)+\cdots \tag{29}$$

and in constructing the wavefunction (17), we are instructed to integrate once and divide by ν, which gives:

$$\begin{aligned}
\mathcal{F} &= \frac{2}{3}\frac{z^{\frac{3}{2}}}{\nu}+\frac{1}{2}\left(\frac{1}{2}+\Gamma\right)\ln z+\frac{1}{48}\frac{\nu^2}{z^{3/2}}(12\Gamma^2+12\Gamma+5)+\cdots\\
&= \frac{2}{3}g_s^{-1}+\frac{1}{2}\left(\frac{1}{2}+\Gamma\right)g_s^0\ln\mu+\frac{1}{48}g_s^1(12\Gamma^2+12\Gamma+5)+\cdots\\
\bar{\mathcal{F}} &= -\frac{2}{3}g_s^{-1}+\frac{1}{2}\left(\frac{1}{2}-\Gamma\right)g_s^0\ln\mu-\frac{1}{48}g_s^1(12\Gamma^2-12\Gamma+5)+\cdots
\end{aligned} \tag{30}$$

The rewriting in terms of powers of $g_s=\frac{\nu}{\mu^{3/2}}$ in the final lines of each equation shows that this is clearly a worldsheet expansion in surfaces with boundary, but there are *two types* of boundary. Those coming from strings ending on ZZ D–branes have a factor of Γ associated to them (as in the analogous $u(z)$ expansion (8)), but there is another type of boundary. It is *not* of ZZ D–brane type, and so it has no factor of Γ associated to them. Such boundaries are present in every worldsheet. They are to be associated to a *single* FZZT D–brane, which

is in the presence of the background as a probe. This sum of connected diagrams (all with a boundary on the FZZT D–brane) is evidently the free energy of the FZZT D–brane in the presence of the Γ background ZZ D–branes. Exponentiation to form the wavefunction ψ is then the construction of the *partition function* of this system[25, 26].

There is also particular geometrical meaning to the point $\lambda = 0$ which we are studying here. The space of scaled eigenvalues, forming a continuum, $\lambda \in [0, \infty)$, is the natural space that arises from the underlying double scaled matrix model. Its connection to the target space of the minimal string theory was emphasized in refs.[27, 25]. The minimal string's most natural coordinate (in the continuum approach) is the Liouville direction, φ, which runs from $-\infty$ to $+\infty$. There is a linear dilaton $\Phi \propto \varphi$, and so the string coupling $g_s = e^{\Phi}$ grows with increasing φ. There are two distinct types of D–brane[3, 1, 2]. The ZZ D–branes are localized in φ, but are in the strong coupling region at $\varphi \to +\infty$. The FZZT D–branes are extended in φ, but dissolve and come to an end at a specific value $\varphi = \varphi_\lambda$. The label λ is used because this value is related to our λ as $\varphi \sim -\ln \lambda$. So λ space is the moduli space of FZZT D–brane positions[13], and at a given λ, the wavefunction $\psi(z)$ is the partition function telling us about the physics of the FZZT D–branes *via* open string worldsheets. So the case $\lambda = 0$, which we've been studying so far, is the extreme case of extending the end of the FZZT D–brane probe all the way up to touch the Γ ZZ D–branes residing at $\varphi = +\infty$. This explains rather nicely the form of the expansion that we obtain from the Painlevé II equation in this situation. We have all possible diagrams which start on the FZZT branes; ones which can end on the background D–branes, and ones which do not. See figure 9. In particular, the leading diagram is just the disc diagram measuring the tension of the probe brane as $\tau_{\rm fzzt} = \frac{2}{3} g_s^{-1}$.

There is also the limit $z \to -\infty$. Expanding $v_{C_\pm}$ in this limit (see equation (19)), dividing by ν and integrating once gives:

$$\mathcal{F} \;=\; -\left(\frac{1}{2} + \Gamma\right) g_s^0 \ln \mu + \frac{1}{24} g_s^2 \left(4\Gamma^2 - 1\right)(2\Gamma + 3) + \cdots \tag{31}$$

$$\bar{\mathcal{F}} \;=\; -\left(\frac{1}{2} - \Gamma\right) g_s^0 \ln \mu - \frac{1}{24} g_s^2 \left(4\Gamma^2 - 1\right)(2\Gamma - 3) + \cdots \tag{32}$$

which, like equation (30) for the large positive z expansion, has a purely open string explanation. The Γ ZZ D–branes have been replaced by background R–R flux, which the FZZT D–brane now probes. The diagrams again all contain one FZZT boundary, and there are also diagrams with a puncture by a vertex operator, each such puncture bringing a factor Γg_s, as in the $u(z)$ expansion for the background given in equation (8).See figure 10. There is a selection rule that

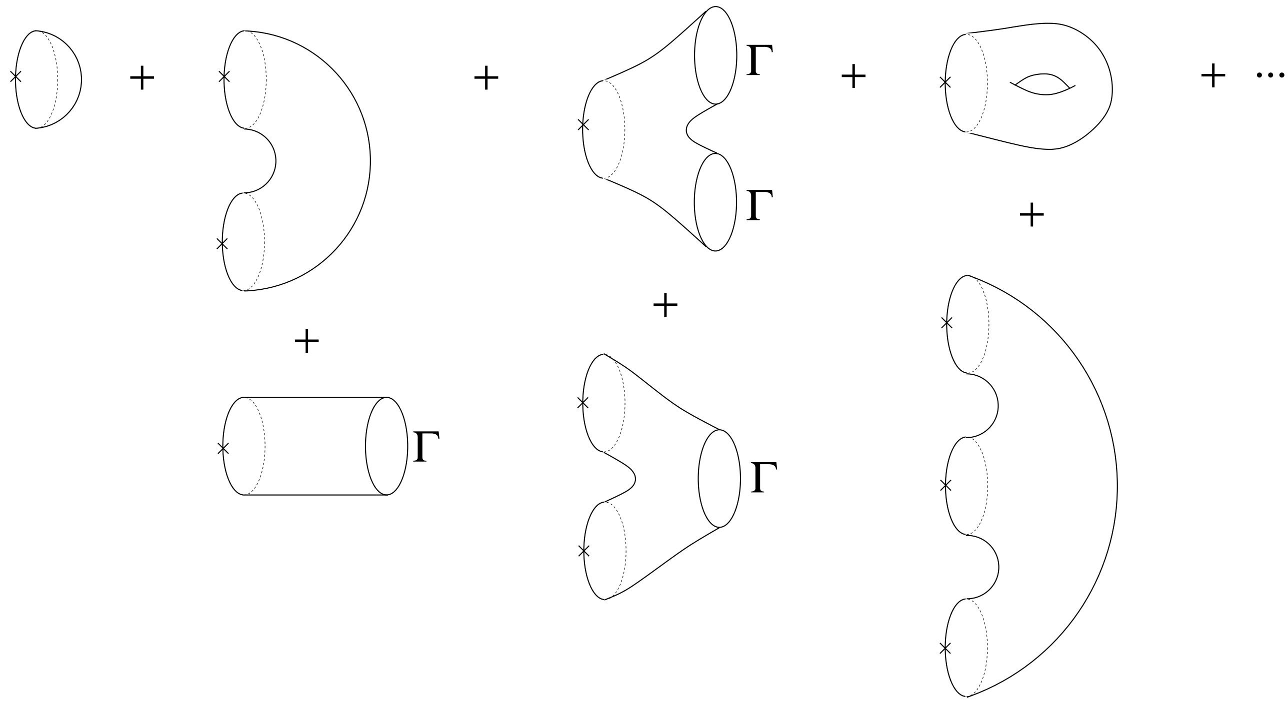


Figure 9: Some diagrams contributing from the worldsheet expansion (30) given by perturbatively solving the Painlevé II equation. for large positive z. There is a single FZZT D–brane probing a background of Γ ZZ D–branes.

the total number of boundaries and punctures, taken together, must be even. This presumably has the same origins as for the similar restriction discussed beneath equation (8).

5 Making Sense of Morons

It is worth remarking here that the appearance of the Painlevé II equation we've noted here –controlling the physics of the fully extended FZZT branes– is different from the manner in which it appears as the physics of the 0B version of this model[13]. In that context, one starts with the string equation (3) for $u(z)$, and uses the transformation of Morris[28] (*i.e.* not the Miura transformation (16)):

$$u(z) = f^2(z) - z \ , \tag{33}$$

to find that *if* $\Gamma = 0$, then $f(z)$ satisfies a Painlevé II equation, with $C = 0$. If $\Gamma \neq 0$, the resulting equation is quite different, as there is an extra term $\nu^2\Gamma^2/f^3$. A further difference is that the type 0B free energy is the second integral of $f^2(z)$, not the first integral of $f(z)$. So on the one hand, these systems are clearly different, and the appearance of Painlevé II in both

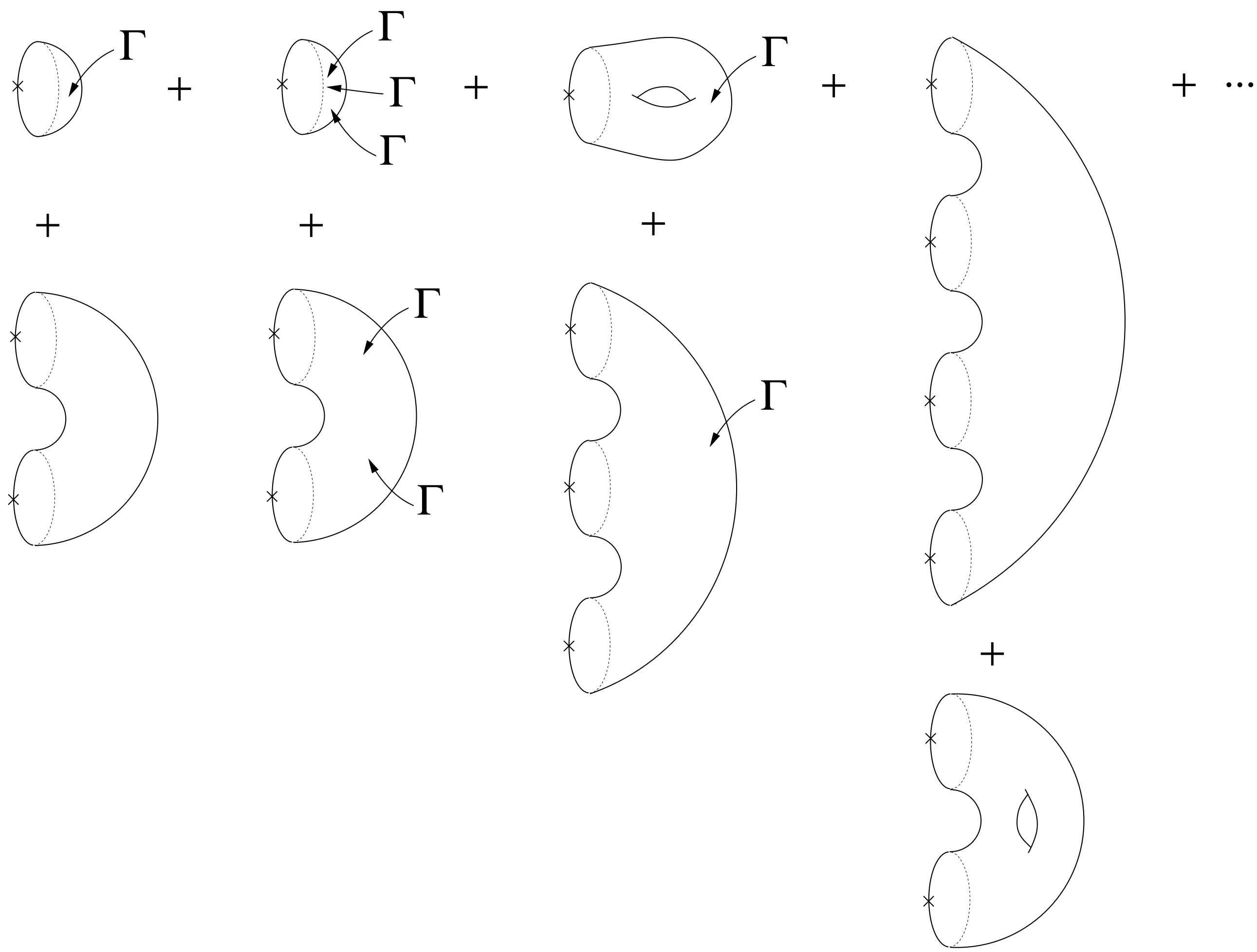


Figure 10: Some diagrams contributing from the worldsheet expansion (32) given by perturbatively solving the Painlevé II equation. for large negative z. There is a single FZZT D–brane probing a background with Γ units of flux.

contexts is a coincidence, but on the other hand, it is an interesting clue as to how to make sense of the cases of $\Gamma = \pm\frac{1}{2}$ here[2]. The functions $v_C(z)$ or $f(z)$ do have interpretations in their own right, the first being a one–point function in the FZZT (with background ZZ D–branes) system, and the second being the square root of a two–point function of the type 0B/0A system. So we should take any relation between them as useful information. When $\Gamma = \pm\frac{1}{2}$, we can have $C = 0$ in our Painlevé II equation determining v_C. In other words, under the Miura map (16),

[2]This is the reason for the title of this subsection. Integer Γ counts the number of D–branes, which are also instantons of the matrix model. An object with half instanton number is called a "meron" in field theory, and so $\Gamma = \pm\frac{1}{2}$, whether it be half a brane or not, deserves the name "moron". (This joke has been waiting in reserve since it arose in a 1996 conversation with Atish Dabholkar, who thought it up in the context of fractional branes in the orientifold models of refs.[29, 30, 31].)

v_C satisfies Painlevé II with $C = 0$, and so we should therefore use the work of ref.[13] and interpret this as the type 0B model with no background D–branes or fluxes! Furthermore, we can use the Morris map (33), to interpret this as a type 0A string again, with no background D–branes or fluxes. So it would seem that there is a role for non–integer Γ after all, but it is subtle: We take $\Gamma = \pm\frac{1}{2}$ here, giving us what would appear to be a poorly behaved 0A/0B theory, but the sector corresponding to an FZZT brane probing this background can be expressed in terms of correlation functions of the well-defined 0A/0B system with no background. This surely deserves to be further explored.

6 General String Equations and Threshold Wavefunctions

In the last sections we focused on the case of the simplest type 0A minimal string model, the $k = 1$ case of the $(2, 4k)$ superconformal series coupled to superLiouville. This was in order to attempt to maintain as clear a presentation of the key facts as possible. Much of what was stated extends naturally to other k, and so it is now worth stating aspects of the overall structure, for completeness.

6.1 The String Equations

The $(2, 4k)$ series of the minimal type 0A string theory, in the presence of background R–R sources, has a non–perturbative definition *via* the following "string equation", which has the structure already shown in equation (3), and repeated here for reference[12]:

$$u\mathcal{R}^2 - \frac{1}{2}\mathcal{R}\mathcal{R}^{''} + \frac{1}{4}(\mathcal{R}^{'})^2 = \nu^2\Gamma^2 \ . \tag{34}$$

The quantity $\mathcal{R}$ is defined by:

$$\mathcal{R} = \sum_{k=0}^{\infty}\left(k + \frac{1}{2}\right) t_k R_k \ , \tag{35}$$

where the R_k $(k = 0, \ldots)$ are polynomials in $u(z)$ and its z–derivatives. They are related by a recursion relation:

$$R^{'}_{k+1} = \frac{1}{4}R^{'''}_k - uR^{'}_k - \frac{1}{2}u'R_k \ , \tag{36}$$

and are fixed by the constant R_0, and the requirement that the rest vanish for vanishing u. The first few are:

$$R_0 = \frac{1}{2} \ ; \quad R_1 = -\frac{1}{4}u \ ; \quad R_2 = \frac{1}{16}(3u^2 - u^{''}) \ . \tag{37}$$

The kth model is chosen by setting all the other ts to zero except $t_0 \equiv -4z$, and t_k, the latter being fixed to a numerical value such that $\mathcal{R} = \mathcal{D}_k - z$. The $\mathcal{D}_k$ are normalised such that the coefficient of u^k is unity, *e.g.*:

$$\mathcal{D}_1 = u \ , \quad \mathcal{D}_2 = u^2 - \frac{1}{3}u^{''} \ , \quad \mathcal{D}_3 = u^3 - uu^{''} - \frac{1}{2}(u^{'})^2 + \frac{1}{10}u^{''''} \ . \tag{38}$$

For the kth model, equation (3) has asymptotics:

$$\begin{aligned} u(z) &= z^{\frac{1}{k}} + \frac{\nu\Gamma}{kz^{1-\frac{1}{2k}}} + \cdots \quad \text{for} \quad z \longrightarrow +\infty \ , \\ u(z) &= \frac{\nu^2(4\Gamma^2 - 1)}{4z^2} + \cdots \quad \text{for} \quad z \longrightarrow -\infty \ . \end{aligned} \tag{39}$$

The function $u(z)$ defines the partition function $Z = \exp(-F)$ of the string theory *via* the equation (6), where μ is the coefficient of the lowest dimension operator in the world–sheet theory. The partition function has perturbative expansions in the dimensionless string coupling

$$g_s = \frac{\nu}{\mu^{1+\frac{1}{2k}}} \ . \tag{40}$$

From the point of view of the kth theory, the other t_ks represent coupling to closed string operators $\mathcal{O}_k$. It is well known that the insertion of each operator can be expressed in terms of the KdV flows[22, 32]:

$$\frac{\partial u}{\partial t_k} = R^{'}_{k+1} \ . \tag{41}$$

6.2 The Wavefunctions and The Painlevé II Hierarchy

In ref.[12], it was established that the double scaled unitary matrix models of refs.[23, 24] had an interpretation as continuum string theories with open string sectors. This was done by establishing a direct connection (using the Miura map (16)) between the solutions of the string equations in equation (3) and solutions of the string equations for those unitary matrix models.

That connection was interesting, and while it was a considerable advance in understanding that double scaled unitary matrix models were intimately related to closed and open string theories obtained by double scaling Hermitian and Complex matrix models, and furthermore represents one of the earliest realizations that open and closed string physics could be related non–perturbatively, it was still not clear exactly how the map fit with a lot of the recent work on minimal strings. For example, as already mentioned, the connection of Complex matrix model formulations of type 0A to string equations for type 0B written as deformations of Painlevé II

were made in ref.[33] using the Morris map (33). Some progress in understanding the results of that paper in a modern context was made in our previous paper[21], where we used the map from one system to another and back to derive the map (the explicit Bäcklund transformation) from u_Γ to $u_{\Gamma\pm1}$. But no explicit role was given to $v_{C_\pm}$ as a natural object in its own right; it was merely a device to facilitate the derivation of the Γ–changing map.

Now we see clearly the role of all of the physics of ref.[12]. At $\lambda = 0$, the wavefunction $\psi(z)$. can always be written in the form given in equation (17), and the function $v_{C_\pm}$ satisfies the equations of the Painlevé II hierarchy. These equations are most naturally written in terms of the quantities S_k, where:

$$S_k \equiv \frac{1}{2}R_k'[v^2 - v'] - vR_k[v^2 - v'] \ . \tag{42}$$

The equations of the Painlevé II hierarchy are

$$\sum_{k=1}^{\infty}\left(k + \frac{1}{2}\right) t_k S_k[v(z)] + zv(z) + \nu C = 0 \ . \tag{43}$$

Let us briefly recall the proof of the map between the equations and its invertibility. We define $u(z)$ and $v(z)$ such that:

$$X_\pm[u, v] \equiv \frac{1}{2}\mathcal{R}'[u] \mp \nu\Gamma - v(z)\mathcal{R}[u] = 0 \ , \tag{44}$$

which implies a specific form for $v(z)$ given a $u(z)$:

$$v = \frac{\frac{1}{2}\mathcal{R}'[u] \mp \nu\Gamma}{\mathcal{R}[u]} \ . \tag{45}$$

Noting the identity[12]:

$$0 = X_\pm^2 \pm \nu\Gamma X_\pm - \mathcal{R}[u]X_\pm' \equiv (v^2 - v')\mathcal{R}^2[u] - \frac{1}{2}\mathcal{R}[u]\mathcal{R}''[u] + \frac{1}{4}(\mathcal{R}'[u])^2 - \nu^2\Gamma^2 \ , \tag{46}$$

we see that if the inverse transformation $u = v^2 - v'$, Miura map (16) exists, then this is just our original string equation (3). On substitution, equation (45) gives equation (43) with $C = 1/2 \pm \Gamma$. Since the unitary matrix models of refs.[23, 24] were originally derived with $C = 0$, those models turn out to be identified with the cases $\Gamma = \pm 1/2$.

Finally, note that the flows between $v(z)$s for different models are organised by the mKdV hierarchy:

$$\frac{\partial v}{\partial t_k} = \frac{1}{2}S_k'[v] \ , \tag{47}$$

which implicitly defines for us flows between wavefunctions of our system, in other words, defining flows between FZZT D–brane partition functions of the different models.

7 The Role of the Bäcklund Transformations

Recall that the Bäcklund transformation for our string equation's solutions $u(z)$ can be derived by combining the Miura map (16) with the sign flip symmetry of the Painlevé II equation (18), resulting in, for example:

$$u_{\Gamma+1} = v_\Gamma^2 + v_\Gamma' \ , \tag{48}$$

$$u_\Gamma = v_\Gamma^2 - v_\Gamma' \ , \tag{49}$$

$$u_{\Gamma-1} = \bar{v}_\Gamma^2 + \bar{v}_\Gamma' \ . \tag{50}$$

Recall that for a given value of Γ, u_Γ is related by the Miura map to the function v_Γ (which solves Painlevé II with a constant $C = \frac{1}{2} + \Gamma$) and also to $\bar{v}_\Gamma$ (which solves Painlevé II with a constant $1 - C = \frac{1}{2} - \Gamma$). From the discussion under equation (23), and as is clear on the diagram in figure 8, we also have the relations:

$$v_\Gamma \quad = \quad -\bar{v}_{\Gamma+1} \ , \tag{51}$$

$$\text{and} \quad \bar{v}_\Gamma \quad = \quad -v_{\Gamma-1} \ , \tag{52}$$

which shall be useful later. Subtracting equation (48) from (49) yields:

$$u_\Gamma + 2v_\Gamma' = u_{\Gamma+1} \ , \tag{53}$$

while the difference of equations (50) and (49) yields:

$$u_\Gamma + 2\bar{v}_\Gamma' = u_{\Gamma-1} \ . \tag{54}$$

It is amusing to see how these exact expressions work on the worldsheet expansions we have discussed previously ((5) for $u(z)$ and (19) for $v(z)$ and $\bar{v}(z)$), and also on the numerical solutions (see figures 5 and 6).

So it is as if adding two ($\lambda = 0$) FZZT branes given by v is equivalent to adding one ZZ brane, while adding two ($\lambda = 0$) FZZT branes given by $\bar{v}$ has the effect of removing one ZZ D–brane (or flux units). This is not strictly correct however, because it neglects the interaction between the two FZZT branes[25]. Another way of looking at things is to use equation (51) and (52) to rewrite equations (53), and (54) as:

$$u_{\Gamma+1} + \bar{v}_{\Gamma+1}' = u_\Gamma + v_\Gamma' \ , \tag{55}$$

so the background with $(\Gamma + 1)$ ZZ branes (or flux units) and one $\bar{v}$–FZZT brane is the same as the background with Γ ZZ branes (or flux units) and one v–FZZT brane.

Recall from the discussion of supersymmetric quantum mechanics that the Miura map arises from factorising $\mathcal{H}$ in the form:

$$\mathcal{H}_\Gamma\psi \equiv (-d^2+u_\Gamma)\psi = (-d+v)(d+v)\psi \tag{56}$$

We found that a solution of this equation is of course just $\psi = e^p$, where $p' = -v$, since function p clearly satisfies $(d+v)e^p = 0$. However, there should also exist another function q also satisfying $\mathcal{H}_\Gamma e^q = 0$ but not $(d+v)e^q = 0$. Writing $e^r = (d+v)e^q$ we see that r satisfies the following two equations:

$$(-d+v)e^r = 0 \tag{57}$$

$$\mathcal{H}_{\Gamma+1}e^r \equiv (d+v)(-d+v)e^r = (-d^2+u_{\Gamma+1})e^r = 0\ . \tag{58}$$

Using (57) we see that $r' = v$ is a solution. Therefore we can write:

$$e^r = (d+v)e^q = (d-r')e^q = (q'-r')e^q\ , \tag{59}$$

which implies the following auto-Bäcklund transformation between q and r:

$$e^{r+q} = q'-r'\ , \tag{60}$$

where q' and r' are both solutions of the Painleve II equation differing by unit Γ. This directly relates solution v_Γ to solutions $v_{\Gamma\pm1}$; and similarly for $\bar{v}$. The Bäcklund transformation (60) is quite cumbersome to use in practice, but we can in fact derive a more explicit version using the same equations that led to the explicit version of the u transformation. We find:

$$\begin{aligned} v_{\Gamma-1} &= -v_\Gamma + \frac{2\nu\Gamma}{\mathcal{R}[v_\Gamma^2 - v_\Gamma']}\ , \quad \bar{v}_{\Gamma-1} = -\bar{v}_\Gamma - \frac{2\nu(\Gamma-1)}{\mathcal{R}[\bar{v}_\Gamma^2 + \bar{v}_\Gamma']} \\ v_{\Gamma+1} &= -v_\Gamma + \frac{2\nu(\Gamma+1)}{\mathcal{R}[v_\Gamma^2 + v_\Gamma']}\ , \quad \bar{v}_{\Gamma+1} = -\bar{v}_\Gamma - \frac{2\nu\Gamma}{\mathcal{R}[\bar{v}_\Gamma^2 - \bar{v}_\Gamma']} \end{aligned} \tag{61}$$

Using the fact that $v_0 = \bar{v}_0$ it is easy to show inductively that $v_\Gamma = \bar{v}_{-\Gamma}$ if and only if Γ is an integer! This is borne out numerically in the same way[3] that we showed $u_\Gamma = u_{-\Gamma}$ for integer Γ (see figures 4, 5, and 6). The procedure is identical to that used in the continuation of Bessel functions, J_n, of non–integer n to those of integer n [21]. Amusingly, we now observe that it fits rather nicely with the fact that in section 2 we noted that the wave equation (56) in the $z \to -\infty$ limit is in fact (after a simple change of variables) Bessel's equation with $n = \Gamma$. So it is not an analogy for $v(z)$ (in that regime) but the *identical* system!

[3]That is, starting with $\Gamma = \epsilon$, Bäcklund transforming to $\Gamma = -1+\epsilon$, and letting $\epsilon \to 0$.

These results are of course consistent:

$$u_\Gamma = v_\Gamma^2 - v'_\Gamma = \bar{v}_\Gamma^2 - \bar{v}'_\Gamma = v_{-\Gamma}^2 - v'_{-\Gamma} = u_{-\Gamma} \tag{62}$$

So, in many ways it is as if v is an FZZT brane and $\bar{v}$ is an anti-FZZT brane! The only way that this is consistent under charge conjugation $\Gamma \to -\Gamma$ (under which the physics should be invariant) is for integer Γ. This is yet more evidence that Γ is quantized.

An interesting observation is that if one adds v_Γ and $\bar{v}_\Gamma$ together asymptotically then one obtains:

$$\begin{aligned} v_\Gamma + \bar{v}_\Gamma &= \frac{\nu}{2z} - \frac{3\nu^2\Gamma}{4z^{5/2}} + \frac{15\nu^3}{32z^4} + \frac{3\nu^4\Gamma^2}{2z^4} - \frac{\nu^5\Gamma(420\,\Gamma^2+459)}{z^{11/2}} + \cdots \qquad z \to \infty \\ v_\Gamma + \bar{v}_\Gamma &= -\frac{\nu}{z} - \frac{\nu^3(24\Gamma^2-3)}{4z^4} - \frac{\nu^5(240\Gamma^4 - 504\Gamma^2+111)}{16z^7} + \cdots \qquad z \to -\infty \end{aligned} \tag{63}$$

This can be interpreted entirely in terms of a background containing just ZZ branes with no FZZT branes. So it is almost as if a brane and and an anti-brane have annihilated in some way. We have again ignored the interaction term between the two branes here, so this 'annihilation' would only be correct if the branes had no interaction with each other. But it is interesting and worthy of note nonetheless.

8 Remarks About τ–functions

Another well–known structure in the theory of integrable systems is the τ–function. The significance of the τ–function in the context of the minimal string theories was noticed long ago in refs.[34, 35], where it was shown that the KdV flows together with the string equation was equivalent to a family of Virasoro constraints on the square root of the "closed" string theory's (what we should now think of as $\Gamma = 0$) partition function. This square root is the τ–function of the KdV hierarchy, and we have

$$Z_{\Gamma=0} = \tau_0^2\ , \qquad L_n\tau_0 = 0\ , \quad n \geq -1\ , \tag{64}$$

where the L_n are given in terms of products and derivatives with respect to the t_k, and hence the Virasoro constraints can be thought of as an infinite family of relationships between correlation functions of the operators to which the t_k couple.

In fact, there is always a natural *pair* of τ–functions for the KdV/mKdV system in general, and one might wonder where the other one fits into our story. Some remarks about this were already made at the end of ref.[12], and using the results of ref.[36], where the complete structure of

the Virasoro constraints (when open string (and flux) sectors are included) for $\Gamma\neq 0$ were presented, together with the insights gained with the results of the present paper, we can complete the story.

The notation that is usually used for the pair is τ_0 and τ_1, but for out purposes it is probably better to, for a given value of Γ, use the notation τ_Γ and $\tau_{\Gamma+1}$, as will become clear. Then, the partition function of our full string theory at Γ is given by the product:

$$Z_u=\tau_\Gamma\tau_\Gamma\ , \tag{65}$$

while the partition function of the FZZT D–brane probe is the ratio:

$$Z_{\rm probe}=\frac{\tau_{\Gamma+1}}{\tau_\Gamma}\ . \tag{66}$$

Let us see how this relates to the functions u_Γ and v_Γ. We take the logarithm and the second derivative of the partition function to obtain u_Γ, and the first derivative for v_Γ:

$$\begin{aligned} u_\Gamma &= 2\frac{\partial^2}{\partial z^2}\log\tau_\Gamma=2\left(\frac{\tau''_\Gamma}{\tau_\Gamma}-\left(\frac{\tau'_\Gamma}{\tau_\Gamma}\right)^2\right)\ . \\ v_\Gamma &= \frac{\partial}{\partial z}\log\frac{\tau_{\Gamma+1}}{\tau_\Gamma}=\left(\frac{\tau'_\Gamma}{\tau_\Gamma}-\frac{\tau'_{\Gamma+1}}{\tau_{\Gamma+1}}\right)\ . \end{aligned} \tag{67}$$

A little algebra then shows that the combinations $v_\Gamma^2\pm v'_\Gamma$ do indeed give the functions u_Γ and $u_{\Gamma+1}$, if the following relation holds:

$$\tau''_{\Gamma+1}\tau_\Gamma-2\tau'_{\Gamma+1}\tau'_\Gamma+\tau_{\Gamma+1}\tau''_\Gamma=0\ . \tag{68}$$

It is equivalent to the writing of another partition function:

$$Z_{\rm v}=\tau_\Gamma\tau_{\Gamma+1}\ , \tag{69}$$

from which the quantity v_Γ^2 may be derived from a second derivative of the associated free energy:

$$v_\Gamma^2=\frac{\partial^2}{\partial z^2}\log\ \tau_\Gamma\tau_{\Gamma+1}\ . \tag{70}$$

This latter partition function is just that of the original double scaled unitary matrix model[23, 24, 37].

Returning to the matter of the Virasoro constraints, it was noted in ref.[12] that in the presence of Γ, the L_0 constraint would be modified by a Γ–dependent constant. While the constant was not known, the difference between L_0 acting on τ_Γ and L_0 acting on $\tau_{\Gamma+1}$ was expected to be

$\frac{C}{2} = \frac{1}{4} \pm \frac{\Gamma}{2}$. The full structure of he Virasoro constraints for arbitrary Γ was worked out in ref.[36], and the L_0 constraint was shown to be

$$\left(L_0 - \frac{\Gamma^2}{4}\right)\tau_\Gamma = 0 \ , \tag{71}$$

which is consistent with the result of ref.[12], since upon considering the L_0 constraint for a neighbouring value of Γ, we get:

$$\frac{(\Gamma \pm 1)^2}{4} - \frac{\Gamma^2}{4} = \frac{1}{4} \pm \frac{\Gamma}{2} = \frac{C}{2} \ . \tag{72}$$

9 Beyond the Threshold

Let us now consider the case of $\lambda \neq 0$, and examine the structure of $\psi(z)$ at finite energy. For this it is harder to get an exact non–linear ordinary differential equation (another deformation of Painlevé II) for, but a route to the perturbative expansions is as follows. Start again with the factorized form:

$$\mathcal{H}_\Gamma\psi = \left(-d^2 + u_\Gamma\right)\psi = \left[\left(-d + v_C\right)\left(d + v_C\right)\right]\psi = 0 \ , \tag{73}$$

but rather than thinking of it as the case of $\lambda = 0$ (as would have followed from equation (15)), now ask that

$$v_C^2 - v_C' = u_\Gamma + \lambda \ , \tag{74}$$

which is an equation for $v_C(z, \lambda)$, and we take our potential $u_\Gamma(z)$ to satisfy the same string equation as before. Since we chose that factored form again, we still recover the wavefunction by exponentiating as in equation (17).

9.1 Large Positive z

Returning to the case $k = 1$, we get for example (as $z \to +\infty$):

$$\begin{aligned} v_C(z) \quad = \quad & z^{1/2} + \frac{1}{2}\frac{\lambda}{z^{1/2}} + \left(\frac{1}{4} + \frac{\Gamma}{2}\right)\frac{\nu}{z} - \frac{1}{8}\frac{\lambda^2}{z^{3/2}} - \frac{\lambda}{4}\left(\Gamma + 1\right)\frac{1}{z^2} \\ & + \frac{1}{32}\left(2\,\lambda^3 - 5 - 12\,\Gamma^2 - 12\,\Gamma\right)\frac{1}{z^{5/2}} + \frac{\lambda^2}{16}\left(4 + 3\Gamma\right)\frac{1}{z^3} \\ & + \frac{5}{128}\lambda(8\Gamma^2 + 16\Gamma + 10 - \lambda^3)\frac{1}{z^{7/2}} \\ & + \left(\frac{1}{2}\Gamma^3 + \frac{3}{4}\Gamma^2 + \frac{23}{32}\Gamma + \frac{15}{64} - \frac{1}{4}\lambda^3 - \frac{5}{32}\lambda^3\Gamma\right)\frac{\nu^2}{z^4} + \cdots \end{aligned} \tag{75}$$

Now without question this expansion is a mess. On the face of it, there is some difficulty in interpreting these terms in the familiar language of string perturbation theory. It is in fact possible, and the result is quite elegant.

A clue to what to expect is the knowledge that λ controls two things. On the one hand it is the coefficient of a boundary length operator, and on the other hand, it is the distance form the tip of the FZZT D–brane probe at $\varphi = \varphi_c \sim -\ln\lambda$ to the Γ ZZ D–branes located at $\varphi = \infty$. So our worldsheet expansion should make sense in these terms.

That λ is a boundary operator can be seen from the fact[32] that the expectation value of loops of length ℓ can be represented in terms of the Hamiltonian as:

$$w(\ell) = \int dz < z|e^{-\ell\mathcal{H}}|z > \ , \tag{76}$$

and so an $\mathcal{H}$–eigenvalue λ gives it a dependence $e^{-\ell\lambda}$. So λ couples to an operator on the boundary which measures loop length. Since it is a boundary operator, one can expect therefore that it will appear in diagrams with boundary at a given genus with any number of insertions on that boundary. So at every order in string perturbation theory, λ should control an infinite number of terms corresponding to summing over all ways the operator can act on each boundary in that diagram. Armed with that clue, a re–examination of the expansion above yields a useful rewriting. For example one infinite set of terms can be written at disc order as follows:

$$z^{1/2}\left(1+\frac{1}{2}\frac{\lambda}{z}-\frac{1}{8}\left(\frac{\lambda}{z}\right)^2+\frac{1}{16}\left(\frac{\lambda}{z}\right)^3+\cdots\right) = z^{1/2}\left(1+\frac{\lambda}{z}\right)^{1/2} = (z+\lambda)^{1/2} \ . \tag{77}$$

We see that remarkably, the series of terms can be resummed into a remarkably simple expression. There is a simple interpretation of this latter result. The result integrates once to give the disc contribution to the free energy of the probe D–brane. For this $k=1$ model, to leading order in string perturbation theory, the parameter z already couples naturally to the boundary length (since in the Hamiltonian, $u(z) \sim z$, and so this follows from equation (76)), and so switching on λ in this case simply renormalizes z additively. We can write the disc contribution to the free energy from this, after integrating once and dividing by ν:

$$F_{(0,1,0)} = \frac{2}{3}\tilde{g}_s^{-1} \ , \qquad \text{where} \quad \tilde{g}_s = \frac{\nu}{(z+\lambda)^{\frac{3}{2}}} \ , \tag{78}$$

where we have introduced the notation $F_{(b,f,h)}$ to denote the contribution to the free energy from a diagram with f FZZT D–branes, b ZZ D–branes and h handles. This resummation turns out to be precisely what happens to all diagrams involving pure FZZT boundaries and

no loops or ZZ boundaries. For example, let us look at the case of 3 FZZT boundaries. After some algebra, the infinite seires of λ contributions can be resummed, with the result:

$$F_{(0,3,0)} \sim \frac{\nu}{(z+\lambda)^{\frac{3}{2}}} \sim \tilde{g}_s \ . \tag{79}$$

The diagram with 1 FZZT D–brane and one handle also gives a result of the same form, with the total contribution being:

$$F_{(0,3,0)} + F_{(0,1,1)} = \frac{5}{48}\tilde{g}_s \ . \tag{80}$$

Turning to diagrams with both FZZT and ZZ boundaries, let us see what happens to the three–string vertices. We get, for the case of 2 FZZT and 1 ZZ, the result:

$$F_{(1,2,0)} = \frac{\nu\Gamma}{4(z+\lambda)z^{1/2}} = \frac{1}{4}\tilde{g}_s\Gamma\left(\frac{z+\lambda}{z}\right)^{\frac{1}{2}} \ , \tag{81}$$

while the case of 1 FZZT and 2 ZZ yields:

$$F_{(2,1,0)} = \frac{\nu\Gamma^2}{4(z+\lambda)^{1/2}z} = \frac{1}{4}\tilde{g}_s\Gamma^2\left(\frac{z+\lambda}{z}\right) \ . \tag{82}$$

The pattern emerging is clear. Surfaces have an extra factor of

$$\left(\frac{z+\lambda}{z}\right)^{\frac{1}{2}}\Gamma \tag{83}$$

for every boundary that ends on a ZZ D–brane. A λ–dependent factor was expected (see above) to appear for every such boundary, in view of the fact that λ sets the separation from the FZZT D–brane's tip to the Γ ZZ D–branes in the background.

Let us consider the structure at higher order in perturbation theory. The contribution from four FZZT boundaries mixes with that of the diagram with two FZZT boundaries and one handle to give, after resumming:

$$F_{(0,4,0)} + F_{(0,2,1)} = \frac{5\nu^2}{128}\frac{d}{dz}\left(\frac{1}{(z+\lambda)^2}\right) = -\frac{5}{64}\tilde{g}_s^2 \ . \tag{84}$$

More interesting are the cases mixing the different types of boundaries. For example, the case of 2 FZZT and 2 ZZ boundaries can be written as:

$$\begin{aligned} F_{(2,2,0)} = \frac{\nu^2\Gamma^2}{8}\frac{d}{dz}\left(\frac{1}{(z+\lambda)z}\right) &= -\frac{\nu^2\Gamma^2}{8}\left[\frac{1}{(z+\lambda)^2 z} + \frac{1}{(z+\lambda)z^2}\right] \\ &= -\frac{\nu^2\Gamma^2}{8}\frac{1}{(z+\lambda)^3}\left[\frac{z+\lambda}{z} + \left(\frac{z+\lambda}{z}\right)^2\right] \\ &= -\frac{\tilde{g}_s^2\Gamma^2}{8}\left(\frac{z+\lambda}{z}\right)\left[1 + \left(\frac{z+\lambda}{z}\right)\right] \ . \end{aligned} \tag{85}$$

The prefactor shows that this result follows our rule, but there are two terms which contribute to the diagram. We expect that this relates to the fact that there are two distinct ways of slicing this diagram to yield the underlying three–string vertices. Some algebra shows that the case of 3 ZZ and 1 FZZT yields:

$$F_{(3,1,0)} = \frac{\nu^2\Gamma^3}{12}\frac{d}{dz}\left(\frac{1}{(z+\lambda)^{1/2}z^{3/2}}\right) \quad = \quad -\frac{\tilde{g}_s^2\Gamma^3}{24}\left(\frac{z+\lambda}{z}\right)^{\frac{3}{2}}\left[1+3\left(\frac{z+\lambda}{z}\right)\right] , \qquad (86)$$

which again has the expected prefactor, and that there is a contribution from two terms fits with the possible decomposition of the diagram.

At the same order in $\tilde{g}_s$ the results for $F_{(1,3,0)}$ and $F_{(1,1,1)}$ mix to give

$$F_{(1,3,0)} + F_{(1,1,1)} = -\frac{5\tilde{g}_s^2\Gamma}{32}\left(\frac{z+\lambda}{z}\right)^{\frac{1}{2}}\left[1+\frac{1}{3}\left(\frac{z+\lambda}{z}\right)+\frac{1}{5}\left(\frac{z+\lambda}{z}\right)^2\right] , \qquad (87)$$

where now there is a third term, fitting with the fact that there is one way of slicing the diagram with a handle to decompose it into three–string vertices.

9.2 Diagrammatics for Large Positive z

In this section we describe in more detail the general structure of the resummations uncovered above. For ease of notation let us define a symbol:

$$(b,f) \equiv z^{-b/2}(z+\lambda)^{-f/2} , \qquad (88)$$

with $(b,f)'$ denoting one differentiation with respect to z, *etc.*, and we'll also use $(b,f)^{(k)}$ to denote k differentiations with respect to z. We have $(b,f)' = -\frac{b}{2}(b+2,f) - \frac{f}{2}(b,f+2)$. In this section we will also define the contribution to the free energy from a particular diagram, $F_{(b,f,h)}$ implicitly to have a factor of $\nu^{-\chi}\Gamma^b$ multiplying it, where $\chi \equiv 2-2h-b-f$ is the Euler number. Once again we will consider the $v_\Gamma(z)$ expansion; although the $\bar{v}_\Gamma(z)$ expansion can be trivially obtained by multiplying the $F_{(b,f,h)}$ by $(-1)^f$.

For a surface with no handles (h=0), we find that its contribution to the free energy in expansion can be resummed (up to an overall constant) into the form:

$$F_{(b,f,0)} \propto (b,f)^{(b+f-3)} \equiv \langle b,f \rangle \qquad (89)$$

If the number of derivatives is negative in (89), as is the case with the disc and the cylinder, then one should integrate instead of differentiating. A similar equation to (89) holds in the

case of h non-zero providing that one or other of b and f is zero instead. We have:

$$\begin{aligned}
F_{(b,0,h)} \propto (b,0)^{(b+h-1)} &\equiv (b+2h,0)^{(b-1)}\,, \quad b,h\neq 0 \\
F_{(0,f,h)} \propto (0,f)^{(f+h-1)} &\equiv (0,f+2h)^{(f-1)}\,, \quad f,h\neq 0 \\
F_{(0,0,h)} &\propto (6h-4,0)^{(-1)}\,.
\end{aligned} \tag{90}$$

When all of b, f and h are non-zero simultaneously then matters become far more complicated. It turns out to be possible to resum every contribution to the free energy as a sum of terms of the form (b,f). The number of terms increases with the 'complexity' of the series, and the type of terms present in each resummation follow a strict and predictable pattern. Below (in equations (91), (92) and (93)) is a list of the first several resummed terms of the $v(z)$ expansion including overall coefficients and written in the standard form of (89) and (90) wherever possible. Recall that is impossible to distinguish between $F_{(b,f,h)}$ and $F_{(b,f-2,h+1)}$ in the expansion. That these terms are a mixture of surfaces can often be seen by the relative lack of simplicity of their coefficients.

$$\begin{aligned}
F''_{(0,1,0)} &= \frac{1}{2}(0,1)\,, \quad F'_{(0,2,0)} = \frac{1}{4}(0,2)\,, \quad F'_{(1,1,0)} = \frac{1}{2}(1,1) \\
F_{(0,3,0)} + F_{(0,1,1)} &= \frac{5}{48}(3,0)\,, \quad F_{(1,2,0)} = \frac{1}{4}(1,2)\,, \quad F_{(2,1,0)} = \frac{1}{4}(2,1) \\
F_{(0,4,0)} + F_{(0,2,1)} &= \frac{5}{128}(0,4)'\,, \quad F_{(1,3,0)} + F_{(1,1,1)} = -\frac{5}{32}(1,5) - \frac{5}{96}(3,3) - \frac{1}{32}(5,1) \\
F_{(2,2,0)} &= \frac{1}{8}(2,2)'\,, \quad F_{(3,1,0)} = \frac{1}{12}(3,1)'
\end{aligned} \tag{91}$$

$$\begin{aligned}
F_{(0,5,0)} + F_{(0,3,1)} + F_{(0,1,2)} &= \frac{221}{16128}(0,5)'' \\
F_{(1,4,0)} + F_{(1,2,1)} &= \frac{15}{64}(1,8) + \frac{5}{64}(3,6) + \frac{3}{64}(5,4) + \frac{5}{128}(7,2) \\
F_{(2,3,0)} + F_{(2,1,1)} &= \frac{25}{128}(2,7) + \frac{5}{32}(4,5) + \frac{23}{192}(6,3) + \frac{7}{64}(8,1) \\
F_{(3,2,0)} &= \frac{1}{24}(3,2)''\,, \quad F_{(4,1,0)} = \frac{1}{48}(4,1)'' \\
F_{(0,6,0)} + F_{(0,4,1)} + F_{(0,2,2)} &= \frac{113}{24576}(0,6)^{(3)}
\end{aligned} \tag{92}$$

$$\begin{aligned}
F_{(1,5,0)} + F_{(1,3,1)} + F_{(1,1,2)} &= -\frac{1105}{2048}(1,11) - \frac{1105}{6144}(3,9) - \frac{221}{2048}(5,7) \\
&- \frac{165}{2048}(7,5) - \frac{385}{6144}(9,3) - \frac{105}{2048}(11,1) \\
F_{(2,4,0)} + F_{(2,2,1)} &= -\frac{15}{32}(2,10) - \frac{45}{128}(4,8) - \frac{69}{256}(6,6) - \frac{29}{128}(8,4) - \frac{7}{32}(10,2) \\
F_{(3,3,0)} + F_{(3,1,1)} &= -\frac{175}{768}(3,9) - \frac{75}{256}(5,7) - \frac{39}{128}(7,5) - \frac{223}{768}(9,3) - \frac{83}{256}(11,1)
\end{aligned} \tag{93}$$

It is obvious from equations (91), (92) and (93) that it is easy to predict the terms that will be present in higher order terms. The coefficients can then be determined by fitting to the form of the expansion. However, we feel that it should also be possible to determine the relative coefficients within any given contribution from some simple underlying set of rules, since the resummations in (91)–(93) are rather messy. By this we mean that, a surface's specific contribution to the free energy should be able to be isolated, up to an overall constant. A natural hypothesis arises by considering the diagrams associated with $\langle b, f\rangle$ as in some way *fundamental* (see figure 11); and then trying to build handled diagrams by *stitching* these fundamental diagrams together by joining boundaries. The obvious guess for how to represent the joining of two surfaces, $\langle b_1, f_1\rangle$ and $\langle b_2, f_2\rangle$, together is multiplication: $\langle b_1, f_1\rangle\langle b_2, f_2\rangle$.

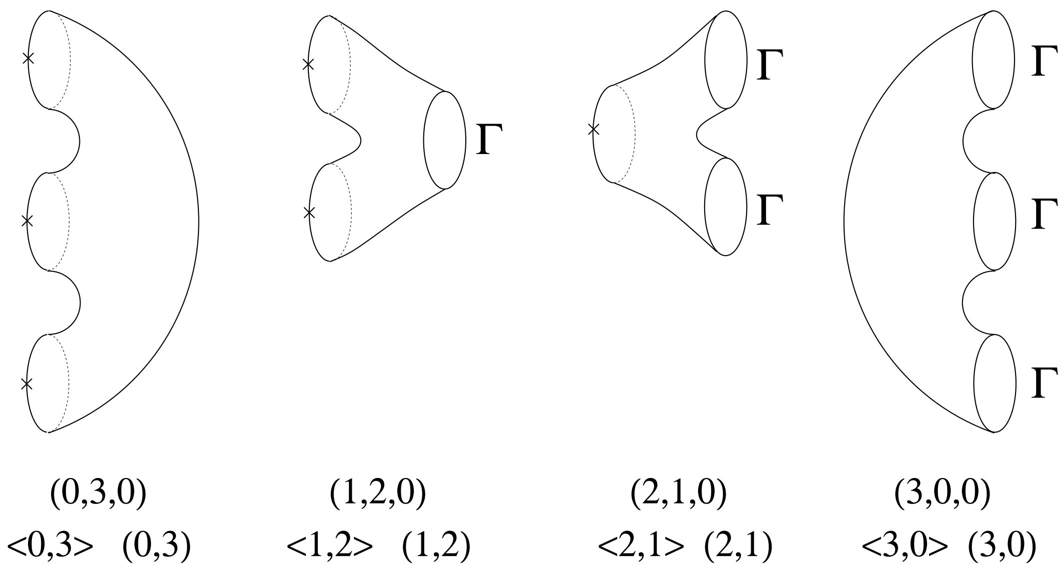


Figure 11: Some of the diagrams fundamental to building diagrams of higher topology (and their contributions the free energy) using the methods described in the text. Shown is some of the notation used to describe them in the text in general ((b, f, h)), to denote their fundamental status ($\langle b, f\rangle$) and also the shorthand for the terms which appear in the free energy ((b, f), see equation (88)).

It is clear from the form of equations (91)—(93) that, for this to work, one must only stitch like boundaries together: that is ZZ boundaries to ZZ boundaries and FZZT boundaries to FZZT boundaries. Notice that if f is zero then surfaces with handles have no λ dependence; and if b is zero then surfaces with handles only depend on the combination $(x + \lambda)$. If these handled terms are formed from the stitching of fundamental surfaces then those with $f = 0$ must be formed exclusively from ZZ–ZZ stitchings; and those with $b = 0$ exclusively from FZZT–FZZT stitchings. Let us explore this by attempting to construct surfaces with $f = 1$.

Since b is arbitrary these surfaces are close in structure to to the $f = 0$ case and so we will try to construct them (at least initially) from ZZ–ZZ stitchings only. Consider the combined $(1, 3, 0)$ and $(1, 1, 1)$ term. Equation (89) tells us the expected form of the former term, and we can denote it as $\langle 1, 3\rangle$. Inspection of figure 12 tells us that the $(1, 1, 1)$ term can only be constructed from the stitching of two pairs of ZZ boundaries between the fundamental three–string surfaces $\langle 3, 0\rangle$ and $\langle 2, 1\rangle$.

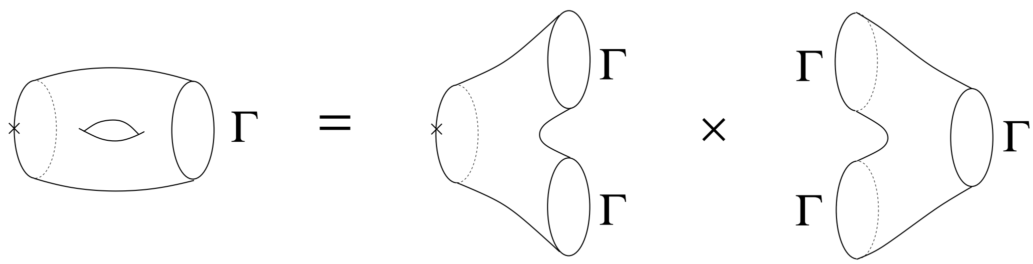


Figure 12: Schematic representation of the stitching of diagrams associated to multiplication to recover terms in the free energy. (See text for details.)

Our guess is therefore:

$$F_{(1,3,0)} + F_{(1,1,1)} = a_0\langle 1, 3\rangle + b_{00}\langle 2, 1\rangle\langle 3, 0\rangle = a_0(1, 3)' + b_{00}(2, 1)(3, 0) \tag{94}$$

Comparison with equations (91) yields the result $a_0 = 5/48$, $b_{00} = -1/32$. Moving on to the combined $(1, 4, 0)$ and $(1, 2, 1)$ term we predict:

$$\begin{aligned} F_{(2,3,0)} + F_{(2,1,1)} &= a_1\langle 2, 3\rangle + b_{10}\langle 2, 1\rangle\langle 4, 0\rangle + b_{11}\langle 3, 0\rangle\langle 3, 1\rangle \\ &= a_2(2, 3)'' + b_{10}(2, 1)(4, 0)' + b_{11}(3, 0)(3, 1)' \end{aligned} \tag{95}$$

The justification for this ansatz is that it contains every pair of constituent surfaces that could possibly contribute. Again, our prediction can be realised with $a_1 = 5/96$, $b_{10} = b_{11} = -1/32$. At the next level we have:

$$\begin{aligned} F_{(3,3,0)} + F_{(3,1,1)} &= a_2\langle 3, 3\rangle + b_{20}\langle 2, 1\rangle\langle 5, 0\rangle + b_{21}\langle 3, 1\rangle\langle 4, 0\rangle + b_{22}\langle 3, 0\rangle\langle 4, 1\rangle \\ &= a_2(3, 3)''' + b_{20}(2, 1)(5, 0)'' + b_{21}(3, 1)'(4, 0)' + b_{22}(3, 0)(4, 1)'' \end{aligned} \tag{96}$$

This time we find that $a_2 = 5/288$, $b_{21} = 2b_{20} = 2b_{22} = -1/32$. We can now see a possible pattern emerging amongst the b_{ij}:

$$b_{ij} = b_i \begin{pmatrix} i \\ j \end{pmatrix} \qquad b_0 = -\frac{1}{32}\,,\ b_1 = -\frac{1}{32}\,,\ b_2 = -\frac{1}{64} \tag{97}$$

Testing this conjecture for the combined $(4,3,0)$ and $(4,1,1)$ term we find that it does indeed hold, with $a_3 = 5/1152$, $b_3 = -1/192$. So it seems that we have uncovered a general rule:

$$F_{(k,3,0)} + F_{(k,1,1)} = a_k\langle k,3\rangle + b_k \sum_{i=0}^{k} \left(\begin{array}{c} k \\ i \end{array} \right) \langle 3+i,0\rangle\langle 2+k-i,1\rangle \tag{98}$$

The origin of the binomial factors here is not known. It may relate to the number of ways that the constituent worldsheets can be stitched together, but we have not been able to how this would work. Notice that upon including the z derivatives in (98), by replacing the $\langle\,,\,\rangle$ with $(\,,\,)$ everywhere, the resultant equation is reminiscent of the Liebniz rule for the multiple differentiation of a product. Note also that it seems that we can now associate the a_i as being the coefficients of the $(i+1,3,0)$ and the b_i as being the coefficients of the $(i+1,1,1)$. Matters are not this simple however, as will be shown when we analyze other surfaces below.

We return now to the combined $(1,4,0)$ and $(1,2,1)$ term. Since the handled term here fits into the $b = 1$ subclass analogous to the $f = 1$ subclass explored above, we make the natural guess that it can be constructed from exclusively FZZT-FZZT stitchings. It turns out that this works providing that we add in an extra term proportional to $(3,2,1)$. This can be thought of as self-stitching of two ZZ boundaries on the worldsheet to form a handle. We find:

$$\begin{aligned} F_{(1,4,0)} + F_{(1,2,1)} &= A_1\langle 1,4\rangle + B_{10}\langle 1,2\rangle\langle 0,4\rangle + B_{11}\langle 0,3\rangle\langle 1,3\rangle + C_1\langle 3,2\rangle \\ &= A_1(1,4)'' + B_{10}(1,2)(0,4)' + B_{11}(0,3)(1,3)' + C_1(3,2)''\ , \end{aligned} \tag{99}$$

where $A_1 = 1/48$, $B_{10} = B_{11} = -1/32$, $C_1 = 1/96$. This seems to be the only sensible choice: all other schemes we tried yielded coefficients as unilluminating as those in the original resummation equations (91), (92) and (93). We can make some sense of this result if consider the action of the transformation $\langle b,f\rangle \leftrightarrow \langle f,b\rangle$ on the resummed contributions. We will denote the action of this transformation on a surface (b,f,h) as $(b,f,h)^\ddagger$. Note that this transformation is defined at the level of the $\langle b,f\rangle$ themselves, so the relation $(b,f,h)^\ddagger \equiv (f,b,h)$ is not trivially satisfied! Indeed, we can immediately see a counter example using (90): $F_{(0,0,h)} \propto (6h-6,0) \neq F_{(0,0,h)^\ddagger} \propto (0,6h-6)$. These $(0,0,h)$ terms are from the background $u(z)$ expansion of course: perhaps terms from the $v(z)$ expansion do satisfy $(b,f,h)^\ddagger \equiv (f,b,h)$? From (89) we clearly see that surfaces without handles certainly do. By requiring that $(1,2,1)^\ddagger = (2,1,1)$ we can unambiguously separate the handled terms from the non-handled terms (using (99) and (95)):

$$\begin{aligned} F_{(2,1,1)} &= \frac{1}{96}\langle 2,3\rangle - \frac{1}{32}\langle 2,1\rangle\langle 4,0\rangle - \frac{1}{32}\langle 3,0\rangle\langle 3,1\rangle \\ F_{(1,2,1)} = F_{(2,1,1)^\ddagger} &= \frac{1}{96}\langle 3,2\rangle - \frac{1}{32}\langle 1,2\rangle\langle 0,4\rangle - \frac{1}{32}\langle 0,3\rangle\langle 1,3\rangle \end{aligned} \tag{100}$$

and, as such:

$$F_{(2,3,0)} = \frac{1}{24}\langle 2,3\rangle\,, \qquad F_{(1,4,0)} = \frac{1}{48}\langle 1,4\rangle \tag{101}$$

This would explain the appearance of the self-stitched $(3,2,1)$ term in (99). Using the same procedure we can render the components of (94) in manifestly invariant form with $(1,1,1)^\ddagger = (1,1,1)$:

$$\begin{aligned} F_{(1,1,1)} = \frac{1}{48}\langle 1,3\rangle &+ \frac{1}{96}\langle 1,2\rangle\langle 2,1\rangle + \frac{1}{48}\langle 3,1\rangle \\ F_{(1,3,0)} &= \frac{1}{12}\langle 1,3\rangle \end{aligned} \tag{102}$$

The $(1,1,1)$ term here now represents two self-stitched surfaces; plus one of each of FZZT-FZZT and ZZ-ZZ stitchings between the fundamental surfaces $\langle 1,2\rangle$ and $\langle 2,1\rangle$.

Whether other surfaces are also invariant under the $b \leftrightarrow f$ operation remains to be seen. At higher orders things get more difficult because of mixing of many different terms with various numbers of handles. The pattern cannot be binomial in every case because for many surfaces there are more possible fundamental stitchings that can contribute to the free energy than there are binomial coefficients. So far we have not been able to complete the elegant picture presented above for arbitrary (b,f,h), though it is likely that the final solution will be extremely simple and symmetrical. This is work in progress. However, if all the surfaces making up $v(z)$ are indeed invariant under $b \leftrightarrow f$, then we can automatically generate the $(1,f,1)$ series from the $(b,1,1)$ series (98). Regrettably, the next term in the $(1,f,1)$ series, $(1,3,1)$, also mixes with a two–handle term, $(1,1,2)$; so without knowing the appropriate two–handle rules we have been unable to test our conjecture.

9.3 The Other Regime: Large Negative z

Let us now turn to the regime $z \to -\infty$. Recall that in the case $\lambda = 0$ we saw that the physics arranged itself into that of a probe FZZT D–brane, now in the background of Γ units of flux. Now we wish to use equation (74) to develop an expansion for $v(z)$ at non–zero λ, and examine its properties. The expansion we obtain is:

$$\begin{aligned} v(z) &= \lambda^{1/2} + \frac{\nu^2(4\,\Gamma^2-1)}{8\,\lambda^{1/2}z^2} - \frac{\nu^3(4\,\Gamma^2-1)}{8\,\lambda z^3} - \frac{\nu^4(16\,\Gamma^4-104\,\Gamma^2+25)}{128\,\lambda^{3/2}z^4} \\ &+ \frac{\nu^4(32\Gamma^4-80\Gamma^2+18)}{32\,\lambda^{1/2}z^5} + \frac{\nu^5(16\Gamma^4-56\Gamma^2+13)}{32\,\lambda^2 z^5} \\ &- \frac{\nu^5(2560\Gamma^4-6400\Gamma^2+1440)}{1024\,\lambda z^6} + \frac{\nu^6(64\Gamma^6-1840\Gamma^4+4768\Gamma^2-1073)}{1024\,\lambda^{5/2}z^6} + \cdots \end{aligned} \tag{103}$$

Once again, we have a remarkable mess, and it is arguably even worse than the one we had for large positive z at non–zero λ. Very interesting is the fact that this expansion does not reduce to the expansion (19) that we had at $\lambda = 0$. In fact, at $\lambda = 0$ if is singular. We shall have to understand the physics of this.

To construct the free energy of our probe (the logarithm of the wavefunction) we are instructed by equation (17) (as we were previously) to integrate once and divide by ν. Having done so, we find that the first few terms are:

$$F = \lambda^{\frac{1}{2}}\frac{z}{\nu} + \frac{(4\Gamma^2-1)}{8}\left(-\frac{\nu}{\lambda^{\frac{1}{2}}z} + \frac{\nu^2}{2\lambda z^2}\right) + \cdots \ , \tag{104}$$

which fits nicely with the asymptotic form of the wavefunction following from modified Bessel functions, observed in equation (14). Now, nowhere do we see a natural occurrence of the dimensionless coupling $g_s = \nu/z^{3/2}$, nor do we see its renormalized cousin $\tilde{g}_s = \nu/(z+\lambda)^{3/2}$, so we appear to have a puzzle.

The resolution is simply that we need not expect either g_s or $\tilde{g}_s$ to appear as the natural stringy expansion parameter in this situation. Part of the reason is that λ and z appear very differently in this regime as compared to the large positive z regime. There, even when λ was zero, at tree level there was a natural parameter which gave Boltzmann weight to loops of length ℓ, and this was z, the tree level part of $u(z)$. Introducing λ brings it in to perform a role already performed by z (at tree level) and so it simply shifts z as we have seen. In this large negative z regime, we have completely different behaviour. The potential $u(z)$ *vanishes* at leading order and so at that order there is nothing weighting the length of loops when $\lambda = 0$. So when λ is non–zero, there is now a weighting parameter at tree level, and so the natural loop expansion that it controls need not have anything to do what that of the $\lambda = 0$ case. Furthermore, it must be disconnected from any expansion developed in the $\lambda = 0$ case since the weight $e^{-\lambda\ell}$ allows loops of infinite length to dominate when $\lambda = 0$ which will not yield a good expansion. An expansion in λ should be expected to be singular there, and this is what we have seen. To handle physics at $\lambda = 0$, we should expect to resum at finite λ to the full $v(z,\lambda)$ and then, setting $\lambda = 0$, develop a new expansion in a different parameter. This is the origin of the second and fourth expressions in equation (19).

Correspondingly, we shall therefore not expect that the natural expansion parameter is g_s or $\tilde{g}_s$. Instead, an examination of the expression for the free energy shows that the natural expansion parameter is:

$$\hat{g}_s = \frac{\nu}{\lambda^{\frac{1}{2}}z} \ . \tag{105}$$

Notice that this is also a dimensionless combination of the important parameters in the problem, a combination which is not available when λ vanishes. Inspired by the success of our exploration of the large positive z regime, we write our free energy as:

$$\begin{aligned} F &= \hat{g}_s^{-1} - \left(\Gamma^2 - \frac{1}{4}\right)\Bigg[\frac{1}{2}\hat{g}_s^1 + \frac{1}{4}\hat{g}_s^2 + \hat{g}_s^3\left(-\frac{4\Gamma^2-25}{96} + \frac{\lambda}{z}\frac{4\Gamma^2-9}{16}\right) \\ &+ \hat{g}_s^4\left(-\frac{4\Gamma^2-13}{32} + \frac{\lambda}{z}\frac{4\Gamma^2-9}{8}\right) \\ &+ \hat{g}_s^5\left(-\frac{(16\Gamma^4-456\Gamma^2+1073)}{1280} - \frac{\lambda}{z}\frac{(4\Gamma^2-9)(4\Gamma^2-61)}{192} + \left(\frac{\lambda}{z}\right)^2\frac{(4\Gamma^2-9)(4\Gamma^2-21)}{32}\right) \\ &+ O(\hat{g}_s^6)\Bigg] \end{aligned} \tag{106}$$

Rather pleasingly, we see that in this new expansion we again have worldsheets corresponding to a probe brane in a background quantified by Γ.

9.4 General Considerations

It is curious that the expansion of $v_C(z)$ manages to resum so nicely, and it is worthwhile to investigate this phenomenon a little more closely. To do so, we again appeal to (74). Let

$$v_C(z) = \sum_{n=0}^{\infty} a_n(z)\nu^n, \qquad u_\Gamma(z) = \sum_{n=0}^{\infty} b_n(z)\nu^n. \tag{107}$$

Note that (5) immediately shows that for $z \to +\infty$,

$$b_0(z) = z, \quad b_1(z) = \frac{\Gamma}{z^{1/2}}, \quad b_2(z) = -\frac{\Gamma^2}{2z^2}, \quad \ldots, \tag{108}$$

and for $z \to -\infty$,

$$\begin{gathered} b_0(z) = 0, \qquad b_1(z) = 0, \qquad b_2(z) = \frac{4\Gamma^2-1}{4z^2}, \\ b_3(z) = 0, \quad b_4(z) = \frac{1}{8}\frac{(4\Gamma^2-1)(4\Gamma^2-9)}{z^5}, \quad \ldots. \end{gathered} \tag{109}$$

It is apparent that for $n > 0$, $b_n(z) = b_n z^{-\frac{3}{2}n+1}$ where the b_n are independent of z. Note that for the negative z expansion, $b_{2n+1} = 0$. We are thus able to write (74) as

$$\sum_{n=1}^{\infty} b_n z^{-\frac{3}{2}n+1}\nu^n + b_0(z) + \lambda = \left(\sum_{n=0}^{\infty} a_n(z)\nu^n\right)^2 - \nu\frac{\partial}{\partial z}\left(\sum_{n=0}^{\infty} a_n(z)\nu^n\right), \tag{110}$$

which must hold for each order of ν. Clearly, this gives $a_0(z) = \sqrt{b_0(z)+\lambda}$. It is straightforward to show that, for $n > 0$,

$$a_n(z) = \frac{1}{2a_0(z)}\left(b_n z^{-\frac{3}{2}n+1} + \frac{\partial}{\partial z}a_{n-1}(z) - \sum_{k=1}^{n-1} a_{n-k}(z)a_k(z)\right). \tag{111}$$

Now, for $z \to +\infty$, $b_0(z) = z$, so $a_0(z) = \sqrt{z+\lambda}$. It is then clear that $a_n(z)$ will take a specific form,

$$a_n(z) = \sum_{i=0}^{N} c_i z^{r_i}(z+\lambda)^{s_i}, \tag{112}$$

where the r_i and s_i are integer or half-integer and the c_i are functions of the b_i. To obtain the contribution to the free energy, we want to divide by ν and integrate once. Based on the rules we introduced above, the contributions to the free energy are manifestly expressions in integer and half–integer powers of z and $(z+\lambda)$, so we would expect these integrals to yield expressions that also have this form. In general, integration will produce expressions with the correct z and $(z+\lambda)$ dependence, but they will also have nontrivial logarithmic dependence. On the outset, this is discouraging because we have developed no rules for producing logarithmic contributions to the free energy. But upon closer investigation we notice an interesting pattern: the b_i conspire in such a way as to exactly cancel the logarthmic dependence in the integral. For example, in the integral of $a_2(z)$, the coefficient of the log term turns out to be $b_1^2 + 2b_2$. But (108) shows that $b_1^2 = -2b_2$, exactly the relationship needed to cancel this coefficient. We have verified explicitly that all log dependence cancels up to $n = 7$.

We turn next to the $z \to -\infty$ expansion. In this regime, $b_0(z) = \lambda$, so $a_0(z) = \sqrt{\lambda}$. Furthermore, $b_{2n+1} = 0$ and $b_{2n} \neq 0$, which suggests that an expression for $a_n(z)$ will inevitably depend on the parity of n. The first few terms are:

$$\begin{aligned} a_0(z) = \sqrt{\lambda}, \quad a_1(z) = 0, \quad a_2(z) = \frac{b_2}{2\sqrt{\lambda}z^2}, \quad a_3(z) = -\frac{b_2}{2\lambda z^3}, \\ a_4(z) = \frac{b_2(6-b_2)}{8\lambda^{3/2}z^4} + \frac{b_4}{2\sqrt{\lambda}z^5}, \quad a_5(z) = \frac{b_2(b_2-3)}{2\lambda^2 z^5} - \frac{5b_4}{4\lambda z^6}. \end{aligned} \tag{113}$$

It is clear that there is a pattern emerging in the powers of z and λ. It can be proved by induction that the general form for $a_n(z)$ is:

$$a_n(z) = \begin{cases} \sum_{i=0}^{\frac{n}{2}-1} c_i z^{-n+i}\lambda^{-\frac{n}{2}+\frac{1}{2}+i} & \text{for n even} \\ \sum_{i=0}^{\frac{n}{2}-\frac{3}{2}} d_i z^{-n+i}\lambda^{-\frac{n}{2}+\frac{1}{2}+i} & \text{for n odd,} \end{cases} \tag{114}$$

where the c_i and d_i are constants to be determined by the b_i. Unlike the case in the positive z expansion, the z–dependence here is trivial so we can easily integrate to obtain each term's

contribution to the free energy. Simplifying, we find that

$$\int dz a_n(z) = \Big(\frac{1}{z\sqrt{\lambda}}\Big)^{n-1}\sum_{i=0}^{N} C_i\Big(\frac{\lambda}{z}\Big)^i \tag{115}$$

where the C_i are again functions of the b_i, and $N = \frac{n}{2} - 1$ if n is even and $N = \frac{n}{2} - \frac{3}{2}$ if n is odd. Setting $\hat{g}_s = \nu/(\lambda^{\frac{1}{2}} z)$ allows us to write the free energy as:

$$F = \sum_{n=0}^{\infty} \tilde{g}_s{}^{n-1} \sum_{i=0}^{N} C_i\Big(\frac{\lambda}{z}\Big)^i \tag{116}$$

It is interesting to see a new stringy perturbative regime arise for non–zero λ, with a string coupling $\hat{g}_s$ that is distinct from the previously identified string couplings, $\tilde{g}_s$.

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